The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Mathematics (Computational and Numerical Methods) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Sketch the functions $f(x) = \tan x$ and $g(x) = \dfrac{1}{x}$ on the same graph for $-2\pi \leq x \leq 2\pi$. From this sketch deduce how many times

$$f(x) \;=\; g(x) \tag{1}$$

for $-2\pi \leq x \leq 2\pi$.

Prove that the first positive x value that satisfies equation (1) lies in the interval $(0.5, 1)$. ***(4 marks)***

(ii) When applying the bisection method to find a root of a function, the tolerance ϵ is an upper bound on the actual error and is given by the following inequality

$$\epsilon \geq \frac{1}{2^n}(b-a), \tag{2}$$

where $n = 1, 2, 3, \ldots$ and (a, b) is the initial interval.

For $\epsilon = 10^{-2}$ and the interval $(0.5, 1)$ find the smallest possible value of n that satisfies inequality (2).

Hence use this value of n to find an approximate solution to equation (1) in the interval $(0.5, 1)$. State your final answer to an accuracy of 2 decimal places. ***(6 marks)***

(iii) Show that finding an approximate value of $\sqrt[3]{7}$ with the Newton-Raphson method leads to a recurrence relation of the form

$$x_{n+1} = \frac{2x_n^3 + 7}{3x_n^2}.$$

(3 marks)

 Question 1 continued on next page

1 (continued)

(iv) Factorise the matrix

$$A = \begin{bmatrix} 2 & 8 & 5 \\ 1 & 6 & 8 \\ 1 & 3 & 2 \end{bmatrix}$$

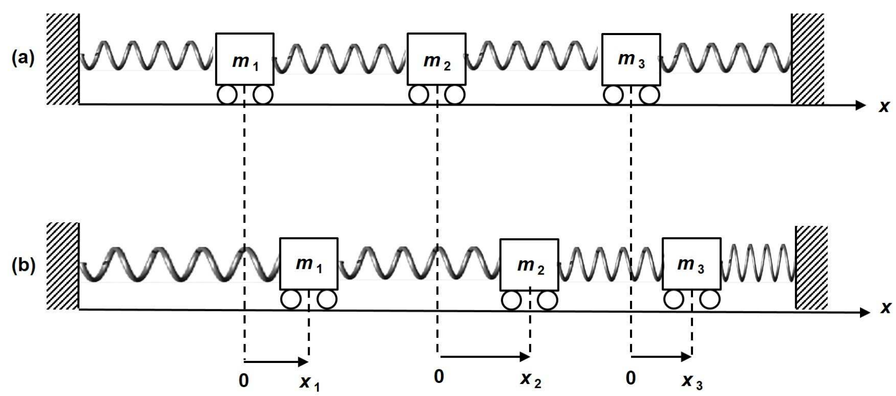
into the product $A = LU$. For this particular example show that $\det A = \det L \det U$. ***(6 marks)***

(v) Use the Jacobi iteration method to find an approximate solution to the following system of equations

$$\begin{aligned} 2x_1 + 10x_2 + 3x_3 &= 19 \\ 3x_1 + 4x_2 + 11x_3 &= 0 \\ 9x_1 + x_2 + x_3 &= 10. \end{aligned}$$

If necessary, first rearrange these equations to ensure convergence. Then starting with the initial column vector $\mathbf{x} = [0, 0, 0]^T$ compute 4 successive iterations, giving your final answer accurate to 3 decimal places. ***(6 marks)***

2 (i) Figure (a) shows three equal masses, m_1, m_2 and m_3, connected by four springs at their equilibrium positions. Figure (b) shows each mass displaced by an amount x_1, x_2 and x_3 respectively.



Assume the motions of the masses are undamped and governed by Hooke's law, i.e., the restoring force of each spring is $F = -kx$, where $k > 0$ is the spring constant and x is the change in spring length from the equilibrium value.

Take the value of k to be equal for all the springs and assume the oscillatory time dependence is $x_1 = X_1 \cos(\omega t)$, $x_2 = X_2 \cos(\omega t)$ and $x_3 = X_3 \cos(\omega t)$, where ω is the angular frequency and X_1, X_2 and X_3 are constants.

2 (continued)

Form the eigenvalue problem of this system and hence, to an accuracy of two decimal places, find all the possible values of ω when $m_1 = m_2 = m_3 = 0.5$ kg and $k = 10$ N m^{-1}.

(12 marks)

(ii) Given the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix},$$

implement the power method to find the *largest* eigenvalue (by magnitude) of matrix A. Do three iterations with the starting eigenvector $[1, 1, 1]^T$ and give your final eigenvalue approximation to an accuracy of three decimal places.

Then by finding the inverse matrix of A, implement the power method to find the *smallest* eigenvalue (by magnitude) of matrix A. Do three iterations with the starting eigenvector $[1, 1, 1]^T$ and give your final eigenvalue approximation to an accuracy of three decimal places.

(13 marks)

3 (i) For the following data

x	0.489	0.524	0.559
y	0.532	0.577	0.625

calculate the interpolated derivative $P_2'(0.532)$ to an accuracy of two decimal places.

Hint: The Lagrange interpolation polynomial of least degree which passes through $(n+1)$ points (x_i, y_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

and $y_i = y(x_i)$.

(5 marks)

 Question 3 continued on next page

3 (continued)

(ii) A theoretical model for the following data predicts exponential behaviour.

x	0.1	0.2	0.3	0.4	0.5
y	0.29	0.33	0.39	0.45	0.53

By employing a suitable transformation, fit an exponential curve to this data by using the least squares linear approximation. Compute the associated coefficients to an accuracy of two decimal places.

Hint: A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a unique polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m \geq n$. Assuming that the x_i values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree n are

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \qquad k = 0, 1, 2, \ldots, n.$$

(15 marks)

(iii) The Taylor series for a function $y(x)$ around a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \ldots$$

Hence, use Taylor series to derive the approximations,

$$y'(x_0) = \frac{y(x_0 + h) - y(x_0 - h)}{2h}$$

and

$$y''(x_0) = \frac{y(x_0 + h) - 2y(x_0) + y(x_0 - h)}{h^2},$$

where $h > 0$ is a constant, showing that the leading terms of the error associated with these approximations are proportional to h^2.

(5 marks)

4 (i) Use the data points (x_0, y_0), (x_1, y_1) and (x_2, y_2) to derive Simpson's rule assuming $x_1 - x_0 = h$ and $x_2 - x_0 = 2h$. ***(10 marks)***

(ii) Using the composite Simpson's rule evaluate

$$\frac{1}{8}\int_1^5 x^3 \ln\left(\sqrt{x}\right)\, dx$$

to an accuracy of $\epsilon = 10^{-3}$. Give your final answer to an accuracy of 3 decimal places.

Hint: If a function $y(x)$ has four continuous derivatives on an interval (a, b) and this interval is divided into n subintervals, where n is an even positive integer, then the error bound for the composite Simpson's rule is given by

$$\left|E_n^S\right| \leq \frac{h^4}{180}(b-a)K,$$

where

$$h = \frac{b-a}{n}$$

and

$$K = \max_{a \leq x \leq b}\left|\frac{d^4 y(x)}{dx^4}\right|.$$

(15 marks)

End of Question Paper