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MAS261

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Further General Engineering Mathematics **3 hours**

This paper has 100 marks. Answer ALL six questions. The questions are weighted differently. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
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to be completed by student

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1 (i) Let

$$f(x,y) = 2x^3 - 6xy + 3y^2.$$

(a) Calculate the directional derivative of $f(x,y)$ at the point $(1,0)$ in the $\mathbf{u} = \mathbf{i} - \mathbf{j}$ direction. **(3 marks)**

(b) Find and classify *all* the critical points of the function $f(x,y)$. **(8 marks)**

(ii) Consider the vector field $\mathbf{F} := (e^y \sin(z), xe^y \sin(z), xe^y \cos(z))$.

(a) Calculate the divergence $\nabla \cdot \mathbf{F}$. **(3 marks)**

(b) Calculate the curl $\nabla \times \mathbf{F}$. **(3 marks)**

(c) Find a scalar potential for $\mathbf{F}$. **(1 mark)**

2 (i) Consider the periodic function $g(t)$ with fundamental period $T = 2$ defined by

$$g(t) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1; \\ 0 & \text{if } -1 < x < 0. \end{cases}$$

Find the Fourier series of $g(t)$. **(11 marks)**

(ii) The Fourier cosine series of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is given by

$$S[\bar{f}](x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(n\pi x).$$

(a) Sketch the graph of $S[\bar{f}](x)$ over the interval $[-2,2]$. **(3 marks)**

(b) Find an exact value for

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(2 marks)

(c) Write down the exponential form of the Fourier cosine series. **(2 marks)**

3 (i) The electrical charge $q(t)$ in a circuit is described by

$$q''(t) + 4q'(t) + 8q(t) = \delta(t)$$

subject to the initial conditions $q(0) = q'(0) = 0$.

(a) Show that the Laplace transform of $q(t)$ is given by

$$Q(s) = \frac{1}{2}\left(\frac{2}{(s+2)^2 + 2^2}\right).$$

(7 marks)

(b) Use the inverse Laplace transform to determine the charge $q(t)$ at time $t > 0$. **(5 marks)**

(ii) Find an explicit formula for $g * k(t)$ when $g(t) = H(t)t^n$ and $k(t) = H(t-1)$. **(4 marks)**

(iii) By considering the Fourier transform of $e^{-|t|}$ or otherwise, find an exact value for the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(\omega)}{1+\omega^2}\, d\omega.$$

(6 marks)

4 (i) Let P, Q, R be mathematical statements. Write down the truth table of the mathematical statement $P \Rightarrow (Q \wedge R)$. **(5 marks)**

(ii) Let (a_n) be the sequence defined by the recurrence relation

$$a_{n+2} \;:=\; 3a_{n+1} - 2a_n$$

with initial conditions $a_1 = 0$ and $a_2 = 1$.

(a) Write down the terms a_3, a_4, a_5 showing all your working (answers with no working will not be given marks). **(3 marks)**

(b) Use mathematical induction to prove that

$$a_n \;=\; 2^{n-1} - 1.$$

(3 marks)

(iii) Let A, B, C be the following sets of positive integers:

$$A := \{p \mid p \text{ is prime}\};$$
$$B := \{2n-1 \mid n \geq 1\};$$
$$C := \{2^n \mid n \geq 2\}.$$

Describe the elements of the set $(A \cap B^c) \cup C$. **(1 mark)**

5 (i) Let $T \subset \mathbb{R}^2$ be the region bounded by the lines $y = x$, $x = 0$, and $y = 1$, and let $f(x, y) = xy$. Find

$$\iint_T f(x,y)dA.$$

(7 marks)

(ii) A cuboid C in $\mathbb{R}^3$ given by

$$C \;:=\; \{(x,y,z) \mid -2 \le x \le 2,\;\; -2 \le y \le 2,\; 0 \le z \le 1\}$$

has density $d(x, y, z) = z(2 - x)(2 - y)$ at the point (x, y, z).

(a) Calculate the mass of the cuboid C. **(4 marks)**

(b) The cylinder $D := \{(x,y,z) \mid x^2 + y^2 \le 1,\; 0 \le z \le 1\}$ is removed from C. What is the remaining mass of $C \backslash D$? **(4 marks)**

6 The temperature $T(x, t)$ in a uniform metal rod of length L satisfies the heat equation

$$\frac{\partial T}{\partial t} \;=\; \kappa \frac{\partial^2 T}{\partial x^2}$$

where $\kappa > 0$ is a constant.

(i) Show that if $T(x, t) = F(x)G(t)$ is a solution to the heat equation then $F(x)$ and $G(t)$ are solutions to the ordinary differential equations

$$F''(x) - \alpha F(x) \;=\; 0 \quad \text{and} \quad G'(t) - \alpha\kappa G(t) = 0$$

where α is an arbitrary constant. **(3 marks)**

(ii) Show that the general solution to the heat equation with boundary conditions

$$\frac{\partial T}{\partial x} = 0 \text{ when } x = 0 \text{ and } \frac{\partial T}{\partial x} = 0 \text{ when } x = L.$$

is given by

$$T(x,t) \;=\; A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)$$

where $A_0, A_1, \ldots$ are arbitrary constants. **(12 marks)**

End of Question Paper

MAS261 FORMULA SHEET

Functions:

- The Heaviside function $H(t)$ is defined by

$$H(t) \;:=\; \begin{cases} 1 & \text{if } t \geq 0; \\ 0 & \text{if } t < 0. \end{cases}$$

- The rectangular function $\mathrm{rect}_T(t)$ is defined by

$$\mathrm{rect}_T(t) \;:=\; \begin{cases} \frac{1}{T} & \text{if } -\frac{T}{2} \leq t \leq \frac{T}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

- The sinc function $\mathrm{sinc}(t)$ is defined by

$$\mathrm{sinc}(t) \;:=\; \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0; \\ 1 & \text{if } t = 0. \end{cases}$$

- The delta function $\delta(t)$ is defined by the property that

$$\int_{-\infty}^{\infty} \delta(t) f(t)\; dt \;=\; f(0) \qquad \text{for all functions } f(t).$$

- The convolution $f * g(t)$ of two functions $f(t)$ and $g(t)$ is defined by

$$f * g(t) \;:=\; \int_{-\infty}^{\infty} f(t-s) g(s)\; ds.$$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\omega_n t) dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\omega_n t) dt.$$

The exponential form of the Fourier series is

$$S[f] \;=\; \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$$

where

$$\omega_n \;=\; \frac{2\pi n}{T}, \qquad c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_n t} dt$$

Laplace transform:

- The Laplace transform of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\ dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad \text{(for } a > 0\text{)}$	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad \text{(for } a > 0\text{)}$	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution
$\mathcal{L}\{tf(t)\} = -F'(s)$ (for $f(t)$ causal)	multiplication by t
$\mathcal{L}\{t^{-1}f(t)\} = \int_s^\infty F(u)du$ (for $f(t)$ causal)	multiplication by t^{-1}
$\mathcal{L}\{\int_0^t f(u)du\} = \frac{1}{s}F(s)$	integration w.r.t. t

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \qquad \text{(for } n \geq 0\text{)}$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s > k$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s > k$
$H(t-T) \qquad \text{(for } T \geq 0\text{)}$	$\frac{e^{-sT}}{s}$	$s > 0$
$\delta(t-T) \qquad \text{(for } T \geq 0\text{)}$	e^{-sT}	$s \in \mathbb{R}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{af(t)+bg(t)\} = a\mathcal{F}\{f(t)\} + b\mathcal{F}\{g(t)\}$	linearity
$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega-\theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\|a\|}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
$e^{-a\|t\|}$ (for $a>0$)	$\frac{2a}{a^2+\omega^2}$
e^{-at^2} (for $a>0$)	$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$
1	$2\pi\delta(\omega)$

The Fourier cosine transform: The Fourier cosine transform of an even function $f(t)$ is given by:

$$\mathcal{F}\{f(t)\}(\omega) \;=\; F(\omega) := \int_0^{\infty} f(t)\cos(\omega t)\;dt$$

and has inverse Fourier cosine transform

$$f(t) \;=\; \frac{2}{\pi}\int_0^{\infty} F(\omega)\cos(\omega t)\;d\omega.$$

The Fourier sine transform: The Fourier sine transform of an odd function $f(t)$ is given by:

$$\mathcal{F}\{f(t)\}(\omega) \;=\; F(\omega) := \int_0^{\infty} f(t)\sin(\omega t)\;dt$$

and has inverse Fourier cosine transform

$$f(t) \;=\; \frac{2}{\pi}\int_0^{\infty} F(\omega)\sin(\omega t)\;d\omega.$$

Integration:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \left(\sqrt{x^2 + y^2}, \arctan(\tfrac{y}{x}), z\right)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \left(\sqrt{x^2 + y^2 + z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\right)$$
$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta.$$

Partial differential equations:

The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(where } c > 0 \text{ is a constant)}$$

and has a general solution

$$u(x,t) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary functions of a single variable.

D'Alembert's solution of the wave equation subject to the initial conditions

$$u(x, 0) = \Phi(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \Psi(x)$$

is given by

$$u(x,t) = \frac{1}{2}\left[\Phi(x - ct) + \Phi(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(z)\, dz.$$