The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** four questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–25; Q2–21; Q3–20; Q4–34)

1 A woman has three shirts. Each day she selects a shirt chosen at random from her collection, independently of what she has worn previously. If she finds she has worn the same shirt for three days in a row (irrespective of when it was last washed) then she decides the shirt needs a wash. A (delayed) renewal process is defined by saying that a renewal occurs whenever she performs a wash.

(a) Let v_n be the probability that a renewal occurs at time n. Explain why $v_1 = v_2 = 0$, and give the value of v_n for $n \geq 3$. Hence show that the generating function $V(s)$, defined as $\sum_{n=0}^{\infty} v_n s^n$ for $|s| < 1$, has the form

$$V(s) = \frac{s^3}{9(1-s)}.$$

(7 marks)

(b) Let u_n be the probability that, given that a renewal occurs at time t, a renewal occurs at time $t+n$. Explain why $u_1 = 1/3$ and $u_2 = 1/9$, and give the value of u_n for $n \geq 3$. Hence show that the generating function $U(s)$, defined as $\sum_{n=0}^{\infty} u_n s^n$ for $|s| < 1$, has the form

$$U(s) = 1 + \frac{1}{3}s + \frac{s^2}{9(1-s)}.$$

(8 marks)

(c) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of days until the first renewal.

(5 marks)

(d) What is the expected time until she performs a second wash? ***(5 marks)***

2 Let (Y_n) be a Markov chain on $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the communicating classes of the Markov chain, and state which states are transient and which are recurrent. ***(6 marks)***

(b) Find all stationary distributions of the Markov chain. ***(7 marks)***

(c) Assume the chain starts at time 0 in state 2.

(i) Prove by induction that, for $n \geq 0$, $P(Y_n = 2) = \left(\frac{1}{4}\right)^n$ and $P(Y_n = 1) = \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)$. ***(6 marks)***

(ii) Assuming that the distribution of Y_n converges to a stationary distribution of the chain, which stationary distribution will that be? ***(2 marks)***

3 A discrete time Markov chain has state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix given by

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the communicating classes of this Markov Chain. ***(4 marks)***

(b) For each state of the Markov Chain, state whether it is recurrent or transient, and give its period. ***(8 marks)***

(c) Find all stationary distributions of the chain. ***(8 marks)***

4 A bee lives in a garden with 5 flowers (labelled A, B, C, D and E). At any time, the bee will either be collecting pollen at one of the flowers, or be in its hive. Following the movement of the bee in discrete time, at each time step the bee, if it is currently at a flower, will return to its hive with probability 1/3, and otherwise will move to one of the other flowers, each with equal probability. If the bee is currently in its hive, at the next time step it will remain in its hive with probability 1/2 and otherwise will pick a flower at random.

Label the possible states of the system as $\{H, A, B, C, D, E\}$ where H represents in the hive and for $i = A, B, C, D, E$ i represents being at flower i, and model the behaviour of the bee as a discrete time Markov chain X_n with state space $\{H, A, B, C, D, E\}$.

(a) Give the transition matrix of the Markov chain. ***(6 marks)***

(b) Show the chain is irreducible and aperiodic. ***(4 marks)***

(c) Explain why the stationary distribution must satisfy $P(X_n = A) = P(X_n = B) = P(X_n = C) = P(X_n = D) = P(X_n = E)$. ***(2 marks)***

(d) Find the unique stationary distribution of the chain. ***(5 marks)***

(e) Let p_i be the probability that the bee will visit the hive before it visits flower C if it is at state i at time 0. Find p_A;
[Hint: Think about whether some of the p_i are equal to one another]
(6 marks)

(f) Find the expected number of steps until the bee reaches flower C

(i) if it is at flower A at time 0;

(ii) if it is in its hive at time 0. ***(7 marks)***

(g) If the bee is currently at flower C, what is the expected number of steps until the bee returns to flower C? ***(4 marks)***

End of Question Paper