1 (i) Temperature is given by \( T = 25 - 4x^2 - y^2 \) in a region about the origin. Sketch contours of \( T \).
Find a vector pointing in the direction of maximum rate of increase of \( T \) at the point \((x, y) = (2, 3)\).
At this point, what is the rate of change of \( T \) in the direction \((1, 1)\)?  
\( \text{(8 marks)} \)

(ii) A flagpole of height \( H \) has mass per unit length given by \( \rho(y) = \sigma/(1 + y) \), where \( y \) is the distance from the ground and \( \sigma \) is a constant. Find the centre of mass of the flagpole. \( \text{Hint: } y = (1 + y) - 1 \).
Determine the moment of the inertia of the flagpole about an axis through its base.  
\( \text{(9 marks)} \)

(iii) If \( r = \sqrt{x^2 + y^2 + z^2} \), then show that \( \partial r / \partial x = x/r \).
A comet of mass \( m_0 \) approaches the Sun of mass \( M \). Show that the potential 
\[ V = -\frac{GMm_0}{r}, \]
defines a central force, where \( r \) is the distance of the comet from the Sun and \( G = 6.67 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2} \).
It is observed that the comet has speed 25 \( \text{km s}^{-1} \) when 500 million km from the Sun. If the mass of the Sun is \( 2 \times 10^{30} \text{kg} \), is it possible for the comet to escape the Sun’s gravitational field?  
\( \text{(8 marks)} \)
Consider the force \( F = (x^2 + y^2) \mathbf{i} + 2xy \mathbf{j} \). Calculate the work done by the force along a straight line path from \( A = (0, 2) \) to \( B = (4, 4) \) by evaluating

\[
W = \int_{A}^{B} F \cdot d\mathbf{r}.
\]

Determine a potential for the force and use it to check your result for the work done by the force going from \( A \) to \( B \).

(13 marks)

A vinyl record may be considered as a uniform laminar of mass \( m \) and radius \( R \). Find an expression for the moment of inertia of the record about the perpendicular axis passing through its centre, \( I_o \), in terms of \( m \) and \( R \).

If the record has a radius of 15 cm and weighs 120 grams, find the torque required to accelerate the record from rest to 33.3 rpm (revolutions per minute) in 2 seconds.

(12 marks)

Consider the scalar

\[
\phi = \frac{\mathbf{b} \cdot \mathbf{r}}{r^n},
\]

where \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \) is a constant vector and \( \mathbf{r} = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \), \( r = |\mathbf{r}| \). Using suffix notation, show that

\[
\frac{\partial \phi}{\partial x_i} = \frac{b_i}{r^n} - \frac{n (\mathbf{b} \cdot \mathbf{r}) x_i}{r^{n+2}}.
\]

(You may assume that \( \partial r/\partial x_i = x_i/r \).)

Find the \( n \) for which \( \phi \) satisfies the Laplace equation.

(11 marks)

State Gauss’s Theorem and show that it holds for the case

\[
\mathbf{F} = x^2 \mathbf{i} - e^z \mathbf{j} + z \mathbf{k},
\]

with the volume bounded by \( x^2 + y^2 \leq a^2 \) and \( 0 \leq z \leq h \).

(14 marks)
(i) Let $\mathbf{u}$ denote the motion of a fluid. Define the material derivative

$$ \frac{D}{Dt} \phi, $$

and describe in a few words what this means for a scalar $\phi$.

The force from an inviscid fluid on a small surface element $dS$ is given by $-p \ dS$, where $p$ is the pressure. The body force due to gravity on a small fluid element of mass $\delta m$ and volume $\delta V$ is $\delta m \ g = \rho \delta V \ g$, where $\rho$ is the density.

Starting from Newton’s second law of motion, derive Euler’s equation of motion for an inviscid fluid. You may use the result that for a scalar $\phi$,

$$ \int_{S} \phi \ dS = \int_{V} \nabla \phi \ dV. $$

(10 marks)

(ii) A flow can be written in terms of a stream function $\psi$, in the form $\mathbf{u} = \nabla \times (\psi \ \hat{z})$. Why must $\mathbf{u}$ represent the flow of an incompressible fluid?

Consider the flow around a stationary cylinder given in cylindrical polar coordinates by

$$ \mathbf{u} = U \left( \frac{\alpha^2}{4 r^2} - 1 \right) \sin \theta \ \hat{r} - U \left( \frac{\alpha^2}{4 r^2} + 1 \right) \cos \theta \ \hat{\theta}, $$

where $\alpha$ is a constant. Use the appropriate boundary condition at the surface of the cylinder to determine the cylinder’s radius, and find the flow far from the cylinder.

Calculate the stream function $\psi$ for this flow.

Explain in a few words how the stream function helps to sketch the flow and produce a sketch for the $\psi$ above. (15 marks)

End of Question Paper
\[ \nabla (\phi + \psi) = \nabla \phi + \nabla \psi \]  
\[ \text{(E.1)} \]

\[ \nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \]  
\[ \text{(E.2)} \]

\[ \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \]  
\[ \text{(E.3)} \]

\[ \nabla (\phi \psi) = \phi \nabla \psi + \psi \nabla \phi \]  
\[ \text{(E.4)} \]

\[ \nabla \cdot (\phi \mathbf{v}) = \phi \nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) \phi \]  
\[ \text{(E.5)} \]

\[ \nabla \times (\phi \mathbf{v}) = \phi \nabla \times \mathbf{v} + \nabla \phi \times \mathbf{v} \]  
\[ \text{(E.6)} \]

\[ \nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \]  
\[ \text{(E.7)} \]

\[ \nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{u} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{u}) \]  
\[ \text{(E.8)} \]

\[ \nabla (\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) \]  
\[ \text{(E.9)} \]

- (E.1-E.3) express the linearity property of the vector operators.
- (E.4-E.7) follow immediately using subscript notation and the product rule. You should know or be able to quickly derive them, e.g.
  \[ \nabla \cdot (\phi \mathbf{v}) = \partial_i (\phi v_i) = \phi \partial_i v_i + v_i \partial_i \phi = \phi \nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) \phi . \]
- (E.8-E.9) you should be able to derive, given the identity
  \[ \epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} . \]
OPERATORS IN CYLINDRICAL POLARS (CPs) AND SPHERICAL POLARS (SPs)

1. CPs (Cylindrical Polars) \( (r, \theta, z) \) \( h_1 = 1, \ h_2 = r, \ h_3 = 1 \)

\[
\nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{\partial V}{\partial z} \hat{z} \tag{CP.1}
\]

\[
\nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_1) + \frac{1}{r} \frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z} \tag{CP.2}
\]

\[
\nabla \times \mathbf{F} = \left[ \frac{1}{r} \frac{\partial F_3}{\partial \theta} - \frac{\partial F_2}{\partial z} \right] \hat{r} + \left[ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial \theta} \right] \hat{\theta} + \left[ \frac{1}{r} \frac{\partial}{\partial r} (r F_2) - \frac{1}{r} \frac{\partial F_1}{\partial \theta} \right] \hat{z} \tag{CP.3}
\]

\[
\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} \tag{CP.4}
\]

2. SPs (Spherical Polars) \( (r, \theta, \phi) \) \( h_1 = 1, \ h_2 = r, \ h_3 = r \sin \theta \)

\[
\nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \tag{SP.1}
\]

\[
\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \{ (\sin \theta) F_2 \} + \frac{1}{r \sin \theta} \frac{\partial F_3}{\partial \phi} \tag{SP.2}
\]

\[
\nabla \times \mathbf{F} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \{ (\sin \theta) F_3 \} - \frac{\partial F_2}{\partial \phi} \right] \hat{r} + \frac{1}{r \sin \theta} \left[ \frac{\partial F_1}{\partial \phi} - \frac{\partial}{\partial r} \{ (r \sin \theta) F_3 \} \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r F_2) - \frac{\partial F_1}{\partial \theta} \right] \hat{\phi} \tag{SP.3}
\]

\[
\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \tag{SP.4}
\]