The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017-2018**

Mechanics and Fluids **2 hours**

Attempt all four questions. The allocation of marks is shown in brackets.

1 (i) Temperature is given by $T = 25 - 4x^2 - y^2$ in a region about the origin. Sketch contours of T.

Find a vector pointing in the direction of maximum rate of increase of T at the point $(x, y) = (2, 3)$.

At this point, what is the rate of change of T in the direction $(1, 1)$?

(8 marks)

(ii) A flagpole of height H has mass per unit length given by $\rho(y) = \sigma/(1+y)$, where y is the distance from the ground and σ is a constant. Find the centre of mass of the flagpole. *Hint:* $y = (1 + y) - 1$.

Determine the moment of the inertia of the flagpole about an axis through its base. ***(9 marks)***

(iii) If $r = \sqrt{x^2 + y^2 + z^2}$, then show that $\partial r/\partial x = x/r$.

A comet of mass m_0 approaches the Sun of mass M. Show that the potential

$$V = -\frac{G\,M\,m_0}{r},$$

defines a central force, where r is the distance of the comet from the Sun and $G = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$.

It is observed that the comet has speed $25\,\mathrm{km\,s}^{-1}$ when 500 million km from the Sun. If the mass of the Sun is 2×10^{30}kg, is it possible for the comet to escape the Sun's gravitational field? ***(8 marks)***

2 (i) Consider the force $\boldsymbol{F} = (x^2 + y^2)\,\boldsymbol{i} + 2xy\,\boldsymbol{j}$. Calculate the work done by the force along *a straight line path* from $A = (0, 2)$ to $B = (4, 4)$ by evaluating $W = \int_A^B \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r}$.

Determine a potential for the force and use it to check your result for the work done by the force going from A to B.

(13 marks)

(ii) A vinyl record may be considered as a uniform laminar of mass m and radius R. Find an expression for the moment of inertia of the record about the perpendicular axis passing through its centre, I_o, in terms of m and R.

If the record has a radius of 15 cm and weighs 120 grams, find the torque required to accelerate the record from rest to 33.3 rpm (revolutions per minute) in 2 seconds.

(12 marks)

3 (i) Consider the scalar

$$\phi = \frac{\boldsymbol{b} \cdot \boldsymbol{r}}{r^n},$$

where $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is a constant vector and $\boldsymbol{r} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, $r = |\boldsymbol{r}|$. *Using suffix notation*, show that

$$\frac{\partial \phi}{\partial x_i} = \frac{b_i}{r^n} - \frac{n\,(\boldsymbol{b} \cdot \boldsymbol{r})\,x_i}{r^{n+2}}.$$

(You may assume that $\partial r/\partial x_i = x_i/r$.)

Find the n for which ϕ satisfies the Laplace equation. ***(11 marks)***

(ii) State Gauss's Theorem and show that it holds for the case

$$\boldsymbol{F} = x^2\boldsymbol{i} - \mathrm{e}^z\boldsymbol{j} + z\,\boldsymbol{k},$$

with the volume bounded by $x^2 + y^2 \leq a^2$ and $0 \leq z \leq h$.

(14 marks)

4 (i) Let $\boldsymbol{u}$ denote the motion of a fluid. Define the material derivative

$$\frac{D}{Dt}\phi \,,$$

and describe in a few words what this means for a scalar ϕ.

The force from an inviscid fluid on a small surface element $\mathbf{d}\boldsymbol{S}$ is given by $-p\,\mathbf{d}\boldsymbol{S}$, where p is the pressure. The body force due to gravity on a small fluid element of mass δm and volume δV is $\delta m\,\boldsymbol{g} = \rho\,\delta V\,\boldsymbol{g}$, where ρ is the density.

Starting from Newton's second law of motion, derive Euler's equation of motion for an inviscid fluid. You may use the result that for a scalar ϕ, $\int_S \phi\,\mathbf{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla}\phi\,\mathrm{d}V$.

(10 marks)

(ii) A flow can be written in terms of a stream function ψ, in the form $\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\,\hat{\boldsymbol{z}})$. Why must $\boldsymbol{u}$ represent the flow of an incompressible fluid?

Consider the flow around a stationary cylinder given in cylindrical polar coordinates by

$$\boldsymbol{u} \;=\; U\left(\frac{\alpha^2}{4\,r^2}-1\right)\sin\theta\,\hat{\boldsymbol{r}} - U\left(\frac{\alpha^2}{4\,r^2}+1\right)\cos\theta\,\hat{\boldsymbol{\theta}}\,,$$

where α is a constant. Use the appropriate boundary condition at the surface of the cylinder to determine the cylinder's radius, and find the flow far from the cylinder.

Calculate the stream function ψ for this flow.

Explain in a few words how the stream function helps to sketch the flow and produce a sketch for the ψ above. ***(15 marks)***

End of Question Paper

VECTOR CALCULUS IDENTITIES

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{E.1}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{E.2}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{E.3}$$

$$\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi \tag{E.4}$$

$$\nabla \cdot (\phi\,\mathbf{v}) = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\phi \tag{E.5}$$

$$\nabla \times (\phi\,\mathbf{v}) = \phi\,\nabla \times \mathbf{v} + \nabla\phi \times \mathbf{v} \tag{E.6}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \tag{E.7}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) \tag{E.8}$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) \tag{E.9}$$

- (E.1-E.3) express the linearity property of the vector operators.
- (E.4-E.7) follow immediately using subscript notation and the product rule. You should know or be able to quickly derive them, e.g.

$$\nabla \cdot (\phi\mathbf{v}) = \partial_i(\phi v_i) = \phi\,\partial_i v_i + v_i \partial_i \phi = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\phi\,.$$

- (E.8-E.9) you should be able to derive, given the identity

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\ .$$

OPERATORS IN CYLINDRICAL POLARS (CPs)
AND SPHERICAL POLARS (SPs)

1. CPs (Cylindrical Polars) (r, θ, z) $h_1 = 1,\ h_2 = r,\ h_3 = 1$

$$\nabla V = \frac{\partial V}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\,\hat{\boldsymbol{z}} \tag{CP.1}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\,\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z} \tag{CP.2}$$

$$\nabla \times \mathbf{F} = \left[\frac{1}{r}\frac{\partial F_3}{\partial \theta} - \frac{\partial F_2}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial r}\right]\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rF_2) - \frac{1}{r}\frac{\partial F_1}{\partial \theta}\right]\hat{\boldsymbol{z}} \tag{CP.3}$$

$$\nabla^2 V = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} \tag{CP.4}$$

2. SPs (Spherical Polars) (r, θ, ϕ) $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

$$\nabla V = \frac{\partial V}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\,\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial \phi}\,\hat{\boldsymbol{\phi}} \tag{SP.1}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \theta}\,\{(\sin\theta)F_2\} + \frac{1}{r\sin\theta}\,\frac{\partial F_3}{\partial \phi} \tag{SP.2}$$

$$\begin{aligned}\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}\,\left[\frac{\partial}{\partial \theta}\{(\sin\theta)F_3\} - \frac{\partial F_2}{\partial \phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r\sin\theta}\left[\frac{\partial F_1}{\partial \phi} - \frac{\partial}{\partial r}\{(r\sin\theta)F_3\}\right]\hat{\boldsymbol{\theta}}\\ +\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2) - \frac{\partial F_1}{\partial \theta}\right]\hat{\boldsymbol{\phi}}\end{aligned} \tag{SP.3}$$

$$\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{SP.4}$$