The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Fluid Mechanics I **2 hours**

Answer all four questions.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Write down the Navier-Stokes (N-S) equation for an incompressible fluid of velocity $\boldsymbol{v}$ with constant density ρ under the influence of the gravitational field, and give a brief explanation of the terms in the N-S equation. State how to recover the Euler equation for the ideal fluid from the N-S equation.

(6 marks)

(ii) Define the Reynolds number by using terms in the N-S equation and state its physical meaning.

(3 marks)

(iii) If $\phi = r^{-1}$ show that $\nabla^2\phi = 0$ for $r \neq 0$, *i.e.* ϕ satisfies Laplace's equation. Here, r is the radial coordinate ($r = \sqrt{x^2 + y^2 + z^2}$).

(6 marks)

(iv) The velocity field of a fluid is given by $\boldsymbol{v}(x, y, z, t) = (t, xz, ty)$ where t denotes time. Calculate the acceleration of a particle.

(5 marks)

(v) From the definition of the vorticity $\omega_i = (\nabla \times \boldsymbol{v})_i$, show that

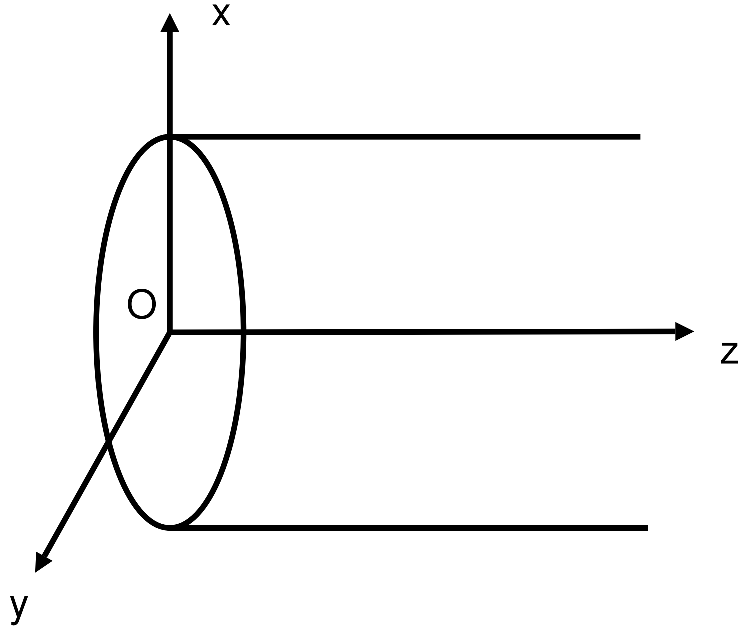
$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = -\epsilon_{ijk}\omega_k.$$

(5 marks)

2 Steady flow of incompressible fluid of density ρ is flowing along an infinitely long fixed horizontal pipe of circular cross-section of radius a. Take axes $Oxyz$ with Oz in the direction of the flow, and you are given that the only non-zero component of velocity is the z-component w, i.e. $\boldsymbol{v} = (0, 0, w)$. There is no gravity (body force) acting on the fluid. You can use this identity

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

where (r, θ, z) represent the cylindrical coordinates.



(i) Show that w is independent of z.

(2 marks)

(ii) Calculate w in terms of the pressure gradient $-P$ where $P > 0$.

(15 marks)

(iii) Calculate the drag per unit length along the cylindrical wall of the tube.

(5 marks)

(iv) Calculate the rate of flow (or volume flux) along the tube.

(3 marks)

3 (i) For a given stress tensor σ_{ij}, write down the ith component of the stress vector $\boldsymbol{t}$ on the surface with the unit normal vector $\boldsymbol{n}$ and explain its physical meaning.

(2 marks)

(ii) Given that $\nabla \cdot \boldsymbol{v} = 0$, show that the stress vector $\boldsymbol{t}$ on a surface element with onward normal $\boldsymbol{n}$ is given by

$$\boldsymbol{t} = -p\boldsymbol{n} + 2\mu(\boldsymbol{n} \cdot \nabla)\boldsymbol{v} + \mu\boldsymbol{n} \times \boldsymbol{\omega}.$$

(10 marks)

(iii) A solid sphere of radius a and center O is moving with constant velocity $\boldsymbol{V}$ in a viscous incompressible fluid, and is not rotating.
Given that

$$\boldsymbol{v} = \left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right)\boldsymbol{V} + \left(\frac{3a}{4r^3} - \frac{3a^3}{4r^5}\right)(\boldsymbol{V} \cdot \boldsymbol{x})\boldsymbol{x}$$

$$p = p_0 + \frac{3\mu a}{2r^3}(\boldsymbol{V} \cdot \boldsymbol{x}),$$

where p_0 is a constant, $\boldsymbol{x}$ is the position vector, and r is the radial distance, verify that the no-slip condition is satisfied. Evaluate the components of the stress tensor on the sphere. You may assume $\nabla \cdot \boldsymbol{V} = 0$ is satisfied.

(13 marks)

4 For an impressible flow with constant density ρ, let the total velocity $\boldsymbol{v}$ and the pressure p consist of the mean and fluctuating parts as follows:

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u},$$

$$p = P + p',$$

where $\langle \boldsymbol{v} \rangle = \boldsymbol{V}$, $\langle \boldsymbol{u} \rangle = 0$, $\langle p \rangle = P$ and $\langle p' \rangle = 0$. Here, the angular brackets denote the average.

(i) Derive the continuity equations for $\boldsymbol{V}$ and $\boldsymbol{u}$. ***(5 marks)***

(ii) Derive the momentum equation for the mean velocity $\boldsymbol{V}$ and give the expression for the Reynolds stress.

(10 marks)

(iii) For a passive scalar field ϕ that is conserved, show that

$$\left\langle \frac{D\phi}{Dt} \right\rangle = \frac{\overline{D}\langle \phi \rangle}{\overline{D}t} + \nabla \cdot \langle \boldsymbol{u}\phi \rangle,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla, \quad \frac{\overline{D}}{\overline{D}t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla.$$

(10 marks)

End of Question Paper