The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2017-2018**

Operations Research **2 hours**

Attempt all FOUR questions.

Please leave this exam paper on your desk
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1 (i) You are given the following linear programming problem (LPP):

$$\max \quad z = 3x_1 - x_2$$

subject to $x_1, x_2 \geq 0$ and

$$2x_1 + x_2 \leq 6, \qquad x_1 - x_2 \leq 1.$$

(a) Find the optimal solution for the problem with the simplex method. Clearly state the optimal solution. ***(10 marks)***

(b) Sketch the feasible region of the LPP, and mark the vertices corresponding to the basic solutions in each tableau. ***(5 marks)***

(ii) The two-phase method is used to solve the LPP:

$$\max \ z = 3x_1 - 5x_2$$

subject to constraints $x_1, x_2 \geq 0$ and

$$x_1 + 2x_2 \leq 7, \quad 4x_1 + 3x_2 = 6, \quad 2x_1 - x_2 \geq 1.$$

Find the initial tableau in phase 1 and process it so that it is suitable to be solved by the simplex method. **Do NOT proceed further.** ***(5 marks)***

(iii) The two-phase method is used to solve an LPP where we maximise

$$z = 4x_1 + 3x_2 - 5x_3$$

subject to conditions $x_1, x_2, x_3 \geq 0$ and two other constraints. The optimal tableau for phase 1 is found as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	Sol
w	0	0	0	0	-1	-1	0
x_2	0	1	1/7	1/7	2/7	$-1/7$	4/7
x_1	1	0	6/7	$-1/7$	5/7	1/7	45/7

where w is the cost function in phase 1, x_4 is the slack variable, x_5 and x_6 are the artificial variables. Find the initial tableau for phase 2 and process it so that it is suitable to be solved by the simplex method. **Do NOT proceed further.** ***(5 marks)***

2 (i) A manufacturer produces two types of goods, $P1$ and $P2$. Both are made from the same raw material. The required labour, raw material, and selling prices of a unit of each product are given as follows:

	Raw material (kg)	labour (hour)	Price (£/unit)
$P1$	5	2	25
$P2$	7	6	45

The availability of raw material and labour is given as follows:

- Raw material can be purchased from a regular provider at a price of £3/kg for the first 200 kilograms, and £2/kg afterwards.
- The provider can supply at most 380 kilograms each week. Additional material has to be purchased from another provider at a cost of £4/kg.
- The manufacturer has 380 hours of free labour available weekly.

Derive a mixed integer-linear programming model for the problem so that the manufacturer can use it to find the optimal production schedule to maximise its weekly profit. **Find the formulation only; do NOT try to find the numerical solution.** Hint: when you define the cost function, you need to take into account the expense of raw material. ***(18 marks)***

(ii) A car manufacturer plans to introduce two new models, M1 and M2. The cars can be assembled using assembly line A alone. Alternatively, one may use a different process, in which each car will go through two assembly lines, line B and line C. **The manufacturer will choose only one of the two methods for the production of all cars**. The time needed to assemble a car on each line and the unit profits are given below:

	A (hour)	B (hour)	C (hour)	Profit (k£/unit)
M1	15	5	7	4
M2	17	4	9	5
Weekly Availability	160	90	100	

Using both lines B and C incurs additional 2k£ operational cost per week.

Formulate a mixed integer linear programming model to find out which method yields higher weekly profit. **Find the formulation only; do NOT try to find the numerical solution.** ***(7 marks)***

3 You are given the following primal linear programming problem

$$\min\ z = c^T x, \quad \text{subject to} \quad A_1 x \leq b_1,\ A_2 x \geq b_2, \text{ and } x \geq 0. \tag{1}$$

(i) Define its Lagrangian function and show that its Lagrangian dual problem can be written as

$$\max\ v = b_2^T y_2 - b_1^T y_1, \tag{2}$$

subject to

$$A_2^T y_2 - A_1^T y_1 \leq c,\ y_1 \geq 0, \text{ and } y_2 \geq 0. \tag{3}$$

(12 marks)

(ii) Define the shadow costs of the primal constraints and find their values in terms of the solutions of the dual problem. ***(4 marks)***

(iii) Show that the dual problem of the problem defined by Equations (2) and (3) is the same as the primal problem in Equation (1). ***(9 marks)***

4 A car maker assembles three different cars on two assembly lines. The models are called F1, F2, F3, and the two assembly lines are line A and line B. All cars go through both assembly lines. Relevant information is given in the table below.

	Processing time (hour)			Yearly capacity (hour)
	F1	F2	F3	
line A	4	10	6	5300
line B	6	8	12	5400
Unit profit (k£)	6	12	10	

To determine the production schedule that maximises the total profit, we define x_1, x_2 and x_3 as the numbers of units of F1, F2, and F3 to be produced, respectively, and formulate the following model:

$$\max \; z = 6x_1 + 12x_2 + 10x_3 \qquad \text{(in k£)}$$

subject to $x_1, x_2, x_3 \geq 0$, and

$$4x_1 + 10x_2 + 6x_3 \leq 5300,$$
$$6x_1 + 8x_2 + 12x_3 \leq 5400.$$

As a first approximation, we solve the problem as a linear programming problem, i.e. we allow non-integer solutions. Introducing slack variables x_4 and x_5 for the first and the second constraints, respectively, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	2/7	6/7	3/7	48000/7
x_2	0	1	-3/7	3/14	-1/7	2550/7
x_1	1	0	18/7	-2/7	5/14	2900/7

(i) Find the optimal cost from the optimal tableau, as well as the optimal solution for the primal variables, and the optimal solution for the dual variables. ***(3 marks)***

(ii) Find the optimality range for the cost coefficient of x_1, and give a brief practical interpretation. ***(6 marks)***

(iii) Suppose that the capacity for assembly line B is reduced to 3000 hours per year due to machine breakdown. Is the original optimal solution still feasible? If not, use the dual simplex method to find the new optimal feasible solution. ***(12 marks)***

(iv) A new model is being considered. It will bring in 8k£ profit per unit; its processing time is 5 hours on assembly line A, and 5 hours on assembly line B too. Is it worth building this car? ***(4 marks)***

End of Question Paper