The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017-2018**

Milestones in Applied Mathematics II **2 Hours**

Answer all four questions.

Please leave this exam paper on your desk
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1 A free particle of mass m is confined to the one-dimensional region $0 \leq x \leq a$. You are given that the orthonormal eigenstate of the Hamiltonian H is

$$\phi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right),$$

where $n = 1, 2, 3.....$

(i) By applying the Hamiltonian H to ϕ_n, show that the energy eigenvalue corresponding to ϕ_n is

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}.$$

(4 marks)

(ii) At $t = 0$, the particle is in the state

$$\psi(x, t = 0) = \frac{3\phi_1 + 4\phi_5}{5}.$$

Show that $\psi(x, t = 0)$ is normalised.

(4 marks)

(iii) Calculate the probability that the particle has the energy E_1 and E_3 at $t = 0$.

(4 marks)

(iv) Calculate the wavefunction at $t > 0$.

(5 marks)

(v) Calculate the expectation values of energy and momentum at $t > 0$.

(8 marks)

2 (i) A person with mass is 70Kg is moving with the speed 6km/hr. Calculate the de Broglie wavelength of this person. You are given that the Planck constant is $h = 6.626 \times 10^{-34}$ J s.

(5 marks)

(ii) The work function of a metal is 2.65×10^{-19}J. What is the threshold frequency of light required to eject electrons from the surface of this metal?

(3 marks)

(iii) For the momentum operator p, calculate the commutator $[e^p, p]$.

(3 marks)

(iv) A particle moving in one dimension interacts with a potential $V(x)$. In a stationary state ψ of this system, show that the expectation value of $x\frac{\partial}{\partial x}V$ in this state ψ is

$$\frac{1}{2}E_\psi\left(x\frac{\partial}{\partial x}V\right) = E_\psi(T),$$

where $T = p^2/2m$ is the kinetic energy of the particle.

(8 marks)

(v) If $\frac{\partial A}{\partial t} = 0$ for any operator A, show that

$$\frac{d}{dt}E_\psi(A) = 0$$

in a stationary state ψ.

(6 marks)

3 A quantum mechanical system is governed by the Hamiltonian

$$H = \hbar\omega\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}.$$

At time $t = 0$, the system is in the state of $\psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(i) Find the energy eigenvalues and corresponding normalised eigenstates.

(10 marks)

(ii) Find the state of the system at $t > 0$.

(9 marks)

(iii) At time t, find the probability of observing the system to be in the state $\psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(6 marks)

4 The simple harmonic oscillator is one of the most important problems in quantum mechanics, which describes the motion of a particle of mass m attached to a spring with a spring constant κ.

(i) Show that the Hamiltonian (energy operator) H for this simple harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Here, p and x are the momentum and position operator respectively; $\omega = \sqrt{\frac{\kappa}{m}}$.

(2 marks)

(ii) Let $m = \omega = \hbar = 1$ for the rest of the question and define an operator A^* as

$$A^* = \sqrt{\frac{1}{2}}\,(x - ip)\,.$$

Calculate the adjoint operator A of A^*.

(2 marks)

(iii) Show that the Hamilton operator can be written as

$$H = A^*A + \frac{1}{2}.$$

(6 marks)

(iv) Hence, show that

$$[H, A] = -A.$$

(6 marks)

(v) You are given that u_n is the energy eigenfunction of H with the energy eigenvalue $E_n = \left(n + \frac{1}{2}\right)$, where $n = 0, 1, 2....$ and $N = A^*A$ is the number operator. In the u_n eigenstate, calculate the expectation value of x and the expectation value of potential energy. ***(9 marks)***

End of Question Paper