



Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

- 1 A function $f(x)$ is defined for $-\infty < x < \infty$ by

$$f(x) = x e^{-x^2}.$$

- (a) Show that for real k the Fourier transform, $\hat{f}(k)$, of $f(x)$ is given by

$$\hat{f}(k) = \frac{1}{2} i \sqrt{\pi} k e^{-k^2/4}. \quad (8 \text{ marks})$$

<p>You may assume that</p> $\int_{-\infty}^{\infty} (x - a) e^{-(x-a)^2} dx = \int_{-\infty}^{\infty} x e^{-x^2} dx$ <p>and</p> $\int_{-\infty}^{\infty} e^{-(x-a)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$ <p>where a may be complex, but does not depend on x.</p>
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- (b) Show that applying the inverse Fourier transform to $\hat{f}(k)$ gives $f(x)$.
(8 marks)
- (c) Verify that Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

holds for this $f(x)$. *(9 marks)*

- 2 The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

- (a) Find, by explicit integration, the Laplace transform of $\cos t$ for $\operatorname{Re} s > 0$.
(5 marks)
- (b) The function $f(t)$ is defined by

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & t > 1. \end{cases}$$

Find the Laplace transform of $f(t)$. (5 marks)

- (c) $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$

Find $F(t)$ for $t > 0$ if $f(t)$ is as defined in part (b), and $g(t) = \cos t$.
(11 marks)

- (d) Given that for $t > 0$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\},$$

use the results of parts (a), (b) and (c) to deduce that the inverse Laplace transform of

$$\frac{1 - e^{-s} - se^{-s}}{s(1 + s^2)}$$

is

$$H(1 - t) - \cos t + H(t - 1)\{\sin(1 - t) + \cos(1 - t)\} \quad \text{for } t > 0,$$

where H is the Heaviside step function. (4 marks)

- 3 The diffusion of a contaminant in a one-dimensional space satisfies the partial differential equation

$$\frac{\partial \Gamma}{\partial t} = \kappa \frac{\partial^2 \Gamma}{\partial x^2} \quad \text{for } -\infty < x < \infty, \quad t \geq 0,$$

where $\Gamma(x, t)$ is the contaminant concentration at position x and time t , and κ is a positive constant.

The Fourier transform of $\Gamma(x, t)$ with respect to x is defined by

$$\hat{\Gamma}(k, t) = \int_{-\infty}^{\infty} e^{ikx} \Gamma(x, t) dx.$$

Show that

$$\hat{\Gamma}(k, t) = \hat{\Gamma}(k, 0)e^{-\kappa k^2 t}. \quad (5 \text{ marks})$$

- (a) The initial condition is

$$\Gamma(x, 0) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & |x| > 1. \end{cases}$$

Find $\hat{\Gamma}(k, 0)$, and show that

$$\Gamma(x, t) = \frac{1}{\sqrt{1 + 4\kappa t}} \exp\left(-\frac{x^2}{1 + 4\kappa t}\right). \quad (12 \text{ marks})$$

$$\left[\begin{array}{l} \text{You may assume that} \\ \int_{-\infty}^{\infty} e^{-b^2(x-a)^2} dx = \frac{1}{b} \sqrt{\pi}, \\ \text{where } b \text{ is real, and } a \text{ may be complex, but does not depend on } x. \end{array} \right]$$

- (b) The initial condition is

$$\Gamma(x, 0) = \sqrt{\pi} \delta(x),$$

where $\delta(x)$ is the Dirac delta function.

Find $\Gamma(x, t)$, and show that as $t \rightarrow \infty$ the answer in (a) gives the same result. (8 marks)

- 4 The function $y(x)$ satisfies the ordinary differential equation

$$x^2 y'' + 3xy' - 3y = x^3 e^{-x}$$

in $0 < x < \infty$, with the boundary conditions that y is finite at $x = 0$ and as $x \rightarrow \infty$.

- (a) By trying $y = x^n$, find the independent solutions of

$$x^2 y'' + 3xy' - 3y = 0. \quad (3 \text{ marks})$$

- (b) Given that Green's function, $G(x; \xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G / \partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x; \xi) = \begin{cases} -\frac{x}{4\xi^2} & 0 \leq x < \xi \\ -\frac{\xi^2}{4x^3} & \xi < x < \infty. \end{cases} \quad (10 \text{ marks})$$

- (c) Using Green's function, show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = \frac{30}{x^3} + e^{-x} \left(x + 5 + \frac{15}{x} + \frac{30}{x^2} + \frac{30}{x^3} \right). \quad (12 \text{ marks})$$

$$\left[\begin{array}{l} \text{You may assume that for } m \geq 1 \\ \int \xi^m e^{-\xi} d\xi = -\xi^m e^{-\xi} + m \int \xi^{m-1} e^{-\xi} d\xi. \end{array} \right]$$

5 Consider the equation

$$(1 - \epsilon)x^3 + 4x^2 + x - 6 = 0, \quad (1)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (1) can be written as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots,$$

where x_0, x_1, x_2, \dots are $O(1)$ as $\epsilon \rightarrow 0$.

Use this expression to show that one of the three solutions to equation (1) is

$$x = -3 - \frac{27}{4}\epsilon + \dots,$$

and to find the other two solutions, correct to order ϵ as $\epsilon \rightarrow 0$.

(14 marks)

(b) Given the rearrangement

$$x = -2 + \epsilon \frac{x^3}{(x-1)(x+3)}$$

of (1), use iteration to find the solution close to -2 , correct to order ϵ^2 as $\epsilon \rightarrow 0$.

(11 marks)

End of Question Paper