Data provided: Formulae sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017-2018

Mathematics (Computational Methods) — Two hours

Attempt all FOUR questions

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Write a Matlab code that could be used to solve the system of equations

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 8 \\
x_1 - 2x_2 + 5x_3 &= 0 \\
4x_1 - 3x_2 + 3x_3 &= 3
\end{aligned}
$$

(2 marks)

(ii) Write a Matlab code that could be used to perform an LU factorisation of the matrix B, where

$$
B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 5 \\ -1 & 5 & -1 \end{pmatrix}
$$

(2 marks)

(iii) (a) Derive the standard five-point difference scheme for the differential equation

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 12xy
$$

at node points $(x_i, y_j) = (ih, jk)$ of a grid with mesh-size $h = k = 0.5$. (4 marks)

(b) Given the above differential equation is satisfied on the rectangular region $0 \le x \le 1.5, 0 \le y \le 1$ together with the boundary conditions

$$
\phi(x, y) = 0 \quad \text{on } x = 0 \text{ for } 0 \le y \le 1,
$$

$$
\phi(x, y) = 3y^3 \quad \text{on } x = 1.5 \text{ for } 0 \le y \le 1,
$$

$$
\phi(x, y) = 0 \text{ on } y = 0 \text{ for } 0 \le x \le 1.5,
$$

and

$$
\frac{\partial \phi}{\partial y} = 6x \text{ on } y = 1 \text{ for } 0 \le x \le 1.5,
$$

derive the equations for the unknown nodal values. You should use a central difference scheme for the derivative boundary condition and draw a diagram with the nodes clearly labelled.

(9 marks)

Write the resulting system comprising four equations and four unknowns in matrix form. Working correct to FOUR decimal places, solve this system of equations using the method of LU decomposition. (8 marks)

2 (i) For the second order partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + 9\frac{\partial^2\phi}{\partial y^2} + 4\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) = 0,$$

where $\phi = \phi(x, y)$ and b is a constant, find the range of values of b such that the differential equation is (a) parabolic, (b) hyperbolic and (c) elliptic. (3 marks)

(ii) (a) Let $a = x_0 < x_1 < \cdots < x_N = b$, and let $f(x_i) = f_i$ for function f which is continuous on $[a, b]$. Describe the properties of the cubic spline interpolant S to f at the equally spaced points $x_i,\ i = 0, 1, \cdots, N$. (3 marks)

(b) Let the cubic interpolant have the form

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

defined on the interval $[x_i, x_{i+1}]$.
Using the substitution $s''(x_i) = \sigma_i \quad (0 \le i \le n)$, where $'$ denotes differentiation with respect to x, derive the relations

$$a_i = \frac{\sigma_{i+1} - \sigma_i}{6h},$$

$$b_i = \frac{\sigma_i}{2},$$

$$c_i = \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1}),$$

$$d_i = f_i,$$

where $h = x_{i+1} - x_i$ and f_i is the value of the function at the i-th position. (8 marks)

(c) Using the conditions derived in (b), determine the natural cubic spline between the following data points

x	0.5	1.0	1.5	2.0
f(x)	1/2	2/3	1.0	3.0

You are given that

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right).$$

You should do your workings either in fractions or correct to FOUR decimal places. (8 marks)

(d) Use the cubic spline interpolant to estimate $f(1.7)$ and $f'(1.7)$. (3 marks)

3 The driving times in minutes between key junctions, A,B,...,L of a road network are given in the table.

Stretch of road between junctions	Driving time in minutes
Stage 0 to Stage 1	
A to B	70
A to C	40
A to D	55
Stage 1 to Stage 2	
B to E	35
B to F	30
C to E	30
C to F	65
C to G	50
D to F	40
D to G	70
Stage 2 to Stage 3	
E to H	40
E to I	60
F to H	40
F to I	70
F to J	45
G to I	40
G to J	25
Stage 3 to Stage 4	
H to L	40
I to L	55
J to L	50

(i) Draw a network diagram to depict the information in the table. (5 marks)

(ii) It is desired to find the quickest route for driving between junctions A and L. Explain how a dynamic programming approach can be used to solve this problem. (4 marks)

(iii) Carry out the process to find the optimal route and the associated minimum travelling time. (16 marks)

4 (i) Find and classify the stationary point of the function

$$f(x, y) = 3x^2 + 2y^2 - 12x + 16y$$

(6 marks)

(ii) Starting at the point $(1, 1)$ perform one iteration of the steepest-descent algorithm to determine an approximation to the minimum point. Do your working to FOUR decimal places. (12 marks)

(iii) Now calculate an approximation to the minimum of $f(x, y)$ using one iteration of Newton's method starting from the point $(1, 1)$. (7 marks)

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$