The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Graph Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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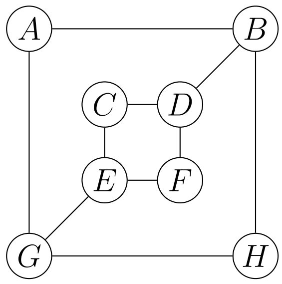
1 The first two parts use the graph Γ shown below.



(i) State what it means for a graph to be semi-Eulerian, and prove that Γ is not semi-Eulerian. List *all* edges that could be added or deleted from Γ to make it semi-Eulerian, while not creating any loops or multiple edges.

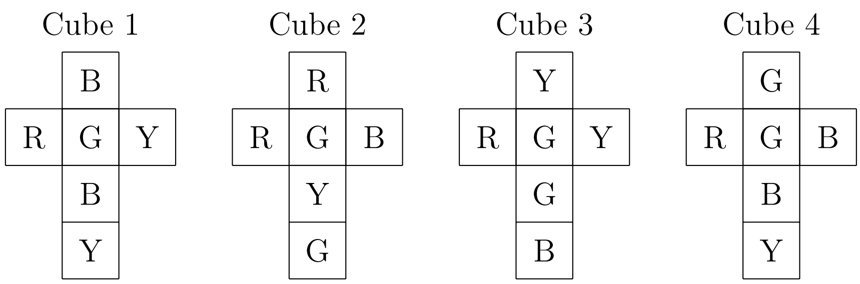
(5 marks)

(ii) State what it means for a graph to be Hamiltonian, and prove that Γ is not Hamiltonian. List *all* edges that could be added or deleted from Γ to make it Hamiltonian.

(5 marks)

(iii) Find a solution to the set of instant insanity cubes shown below. How many other solutions can you get by keeping the position of Cube 1 fixed, but rotating the remaining three cubes? Explain your answer.

(9 marks)



(iv) Recall that in biochemistry, atoms of carbon, nitrogen, oxygen and hydrogen have valency 4, 3, 2 and 1, respectively. Prove that any isotope of C_2NOH_7 is a tree. How many such isotopes are there? Draw them.

(6 marks)

2 The first four parts of this question use a weighted complete graph H on seven vertices $A - G$. The weights of the edges are given below.

A						
20	B					
4	11	C				
6	14	5	D			
19	10	13	15	E		
10	17	11	19	9	F	
15	8	14	7	12	13	G

(i) List the edges of a cheapest spanning tree of H in the order they are added if the tree is built using Kruskal's algorithm.

(2 marks)

(ii) List the edges of a cheapest spanning tree of H in the order they are added if the tree is built using Prim's algorithm beginning at vertex B.

(2 marks)

(iii) Suppose the vertices of H are cities and the edge weights are costs to travel between cities on a direct flight. If journeys with multiple legs are allowed (for instance, getting to F by first going through B), which city is the most expensive to fly to from city A?

(5 marks)

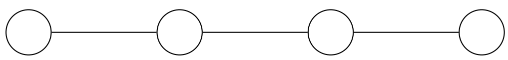
(iv) State the traveling salesperson problem. Give a solution to the traveling salesperson for H, proving that your solution is optimal.

(6 marks)

(v) Draw the tree with Prüfer code 331668.

(4 marks)

(vi) Recall that the path graph P_n consists of n vertices in a line. P_4 is shown below.



How can you tell from the Prüfer code whether or not the corresponding tree is the path graph P_n with some labelling? Explain your answer.

(6 marks)

3 (i) Define the chromatic number, $\chi(G)$, the chromatic index $\chi'(G)$, and the chromatic polynomial $\chi_G(k)$. Explain how to determine $\chi(G)$ from $\chi_G(k)$, and give an example of two graphs G and H whose chromatic polynomials are equal, but whose chromatic indexes are not equal.

(6 marks)

(ii) Prove that for $k \geq 1$ we have $\chi_G(k+1) \geq \chi_G(k)$. Give an example of a G and a $k \geq 1$, with $\chi_G(k+1) = \chi_G(k)$.

(4 marks)

(iii) Prove that if every vertex of G has degree at most d, then $\chi(G) \leq d+1$.

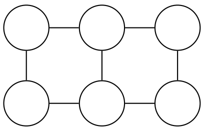
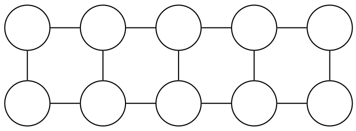
(3 marks)

(iv) Let G and Γ be two subgraphs of H, with the union of G and Γ being all of H, but the intersection $G \cap \Gamma$ consisting of two vertices v and w connected by an edge e. Prove that

$$k(k-1)\chi_H(k) = \chi_G(k)\chi_\Gamma(k)$$

(4 marks)

(v) Let $n \geq 0$. The "ladder graph" L_n has $2n+2$ vertices arranged as n squares in a line. For example, L_1 is isomorphic to the 4-cycle C_4, and L_2 and L_4 are shown below:

L_2  L_4 

Determine the chromatic number, the chromatic index, and the chromatic polynomial of L_n. Justify your answers.

(8 marks)

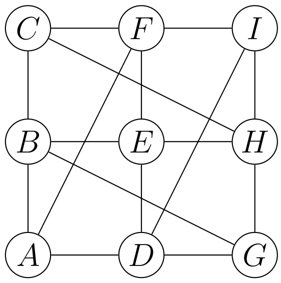
4 (i) Describe the Planarity Algorithm for determining whether or not Hamiltonian graphs are planar, and explain why it works. Using the Planarity Algorithm, prove that $K_{3,3}$ and K_5 aren't planar.

(5 marks)

(ii) State Kuratowski's theorem and prove the "easy" direction, that gives a way to prove graphs aren't planar. You may use Part (i) even if you didn't complete it.

(4 marks)

Parts (iii) and (iv) of this question use the graph Γ shown below.



(iii) Prove Γ is not planar. Draw Γ on the torus, with the vertices labeled.

(5 marks)

(iv) Prove that we can delete any edge e of Γ and the resulting graph $\Gamma \setminus e$ is still not planar. If e is the edge EH, draw $\Gamma \setminus e$ on the Möbius band, with the vertices labeled.

(5 marks)

(v) State Euler's theorem for graphs drawn on the sphere. Use it to prove the following theorem: If a regular graph G of degree 4 is drawn on the sphere so that every face is a triangle, quadrilateral, or pentagon (i.e., every face has 3,4 or 5 edges), then there are exactly 8 more triangles than pentagons.

(6 marks)

End of Question Paper