The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017-2018**

Applicable Analysis **2 hours 30 minutes**

Attempt all four questions. Total marks: 100.

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^{\alpha}e^{bt} (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) (a) Define what is meant by each of the following statements:

(α) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ exists or does not exist, respectively;

(3 marks)

(β) $\displaystyle\int_{a}^{\infty} g(x)\,dx$ exists or does not exist, respectively.

(3 marks)

(b) Prove, **from your definitions,** each of the following statements:

(α) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\,dx$ exists; ***(3 marks)***

(β) $\displaystyle\int_{1}^{\infty} \frac{1}{x}\,(\ln x)^2\,dx$ does not exist. ***(3 marks)***

(ii) (a) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\displaystyle\int_{a}^{\infty} f(x)\,dx$. Your statement should include conditions under which the results are valid. ***(5 marks)***

(b) Decide whether each of the following integrals exists, stating any standard results you need to use:

(α) $\displaystyle\int_{0}^{\infty} \frac{1}{(1+x^{11})^{1/11}}\,dx\,;$ ***(4 marks)***

(β) $\displaystyle\int_{0}^{1} \frac{e^x \sin x}{\sqrt{1-x}}\,dx\,.$ ***(4 marks)***

2 (i) (a) Write down the definition of the Γ function. ***(2 marks)***

(b) Evaluate

(α) $\displaystyle\int_0^\infty \sqrt{y}e^{-y^3}\,dy\,.$ ***(5 marks)***

(β) $\displaystyle\int_1^\infty \frac{\sqrt{\ln y}}{y^2}\,dy.$ ***(3 marks)***

(ii) (a) Write down the definition of the Beta function in terms of an integral over $[0,1]$. ***(2 marks)***

(b) Show that

$$B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du.$$ ***(3 marks)***

(c) Prove that

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta.$$ ***(3 marks)***

(d) Evaluate

$$\int_{-\infty}^\infty \frac{e^{4x}}{(e^{6x}+1)^2}\,dx.$$ ***(7 marks)***

3 (i) In each of the following cases, find the function continuous on $[0, \infty)$, with the given Laplace transform:

(a) $\dfrac{3}{s^2 - 9} \quad (s > 3);$

(b) $\dfrac{s}{s^2 + 2s + 10} \quad (s > 1);$

(c) $\dfrac{2s}{(s^2 + 4)^2} \quad (s > 0).$ ***(6 marks)***

(ii) (a) Let $f : [0, \infty) \to \mathbb{R}$ be continuous and suppose that the Laplace transform $F = L(f)$ exists on (c, ∞) for some $c \in \mathbb{R}$. State, without proof, the formula giving $L\left(\dfrac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

(b) Show that

$$L\left(\frac{1 - e^t}{t}\right) = \ln\left(\frac{s-1}{s}\right) \qquad (s > 1).$$ ***(6 marks)***

(iii) (a) Let $b > 0$. Verify that

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + b^2}\,dx = \frac{\pi\sqrt{2}}{2\sqrt{b}}.$$ ***(5 marks)***

(b) By considering $\displaystyle\int_0^\infty \frac{\sin(xt)}{\sqrt{x}}\,dx$, show that

$$\int_0^\infty \frac{\sin(2x)}{\sqrt{x}}\,dx = \frac{1}{2}\sqrt{\pi}.$$ ***(6 marks)***

4 (i) (a) Suppose the functions f and g are continuous on $[0, \infty)$. Define the convolution $f * g$.

(b) State, without proof, a relation between $L(f * g)$, $L(f)$ and $L(g)$.

(3 marks)

(c) Find the function f continuous on $[0, \infty)$ such that

$$f(t) + \int_0^t f(u)(t-u)\,du = 2\cos t \qquad (t \geq 0).$$

(9 marks)

(ii) Using Laplace transforms, solve the differential equation

$$t\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^{-t}$$

subject to the initial conditions $y(0) = 1$ and $y(1) = \dfrac{2}{e}$. ***(13 marks)***

End of Question Paper