The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2017–2018**

Knots and Surfaces — **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. Strings, pipe cleaners, shoe laces or similar aids for making knots may be used.

Please leave this exam paper on your desk
Do not remove it from the hall

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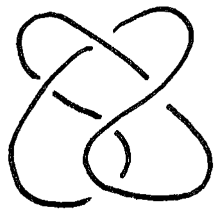
1 (i) (a) State the three Reidemeister moves RI, RII, RIII on knot diagrams. **(7 marks)**

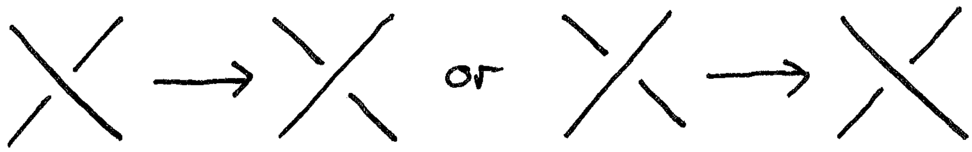
(b) Show with an explicit sequence of Reidemeister moves that the knot diagram below is Reidemeister equivalent to the unknot.



(4 marks)

(c) State Reidemeister's Theorem. **(3 marks)**

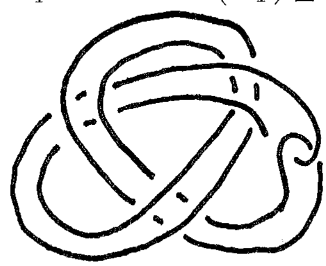
(ii) A crossing change interchanges the under and over strands of a crossing on a link diagram as shown below:



Define the *unknotting number* $u(D)$ of a knot diagram D to be the minimum number of crossing changes required to turn D into a diagram of the unknot, and define the *unknotting number* $u(K)$ of a knot K with knot diagram D_K by

$$u(K) \ := \ \min\{u(D) \mid D \text{ is Reidemeister equvalent to } D_K\}.$$

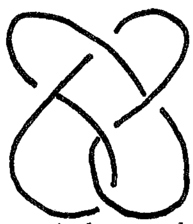
(a) Show that the unknotting number of the Whitehead double of the righthand trefoil W_T below has $u(W_T) \leq 1$.



W_T

(3 marks)

(b) Find the unknotting number of the three-twist knot τ below.



τ

Hint: The three-twist knot is knotted. **(4 marks)**

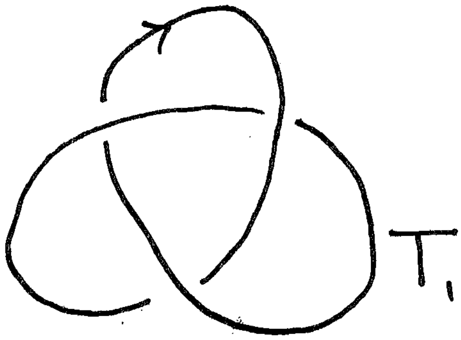
(c) Determine, with a proof or a counterexample, whether the following statement is true:
If D is a diagram with n crossings of a knot K then $u(K) \leq \frac{n}{2}$.
Hint: Descending diagrams are unknotted. **(4 marks)**

2 (i) (a) Define the Kauffman bracket of an unoriented link diagram. **(4 marks)**

(b) Define the writhe of an oriented link diagram, and write down a formula for the Jones polynomial of an oriented link in terms of Kauffman bracket and writhe. **(4 marks)**

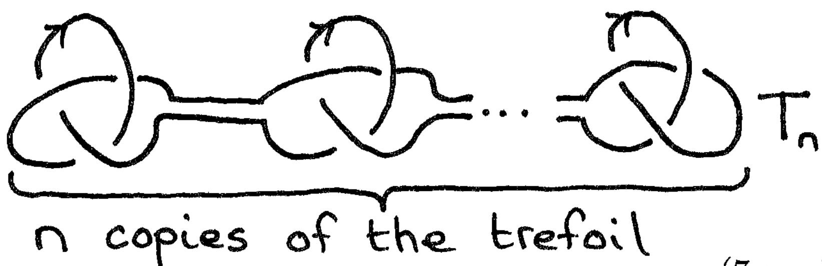
(c) Recall that the reverse of an oriented knot K, is the knot K taken with the opposite orientation and is denoted K_r. Explain why the Jones polynomials $f[K]$ and $f[K_r]$ are equal for every oriented knot K. **(3 marks)**

(ii) (a) Show that the Jones polynomial of the oriented right handed trefoil knot T_1 (shown below) is equal to $-A^{-16} + A^{-12} + A^{-4}$.



(5 marks)

(b) The *connected sum* of n oriented right-handed trefoils T_n is shown below. Using induction or otherwise, show that $f[T_n] = (f[T_1])^n$ where $f[K]$ is the Jones polynomial of K.



(7 marks)

(c) Show that there are infinitely many distinct knots. **(2 marks)**

3 (i) For each of the following subsets of $\mathbb{R}^3$, decide whether it is a surface and decide whether it is compact:

$$\begin{aligned} A &= \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\} \\ B &= \{(x,y,z) \mid z = x^2 - y^2\} \\ C &= \{(x,y,z) \mid x^2+4y^2+9z^2 = 1\} \\ D &= \{(x,y,z) \mid y = xy\}. \end{aligned}$$

No justification of your answers is required. **(8 marks)**

(ii) Draw plane models for the torus and projective plane, and write down surface words for the surfaces. **(4 marks)**

(iii) (a) Define the *non-orientable handle move* between surface words. **(2 marks)**

(b) Draw a sequence of diagrams to indicate why the surfaces associated to the surface words from Part (a) are homeomorphic. **(5 marks)**

(c) Show that the surface word $aabb$ is word equivalent to the surface word $abab^{-1}$. **(4 marks)**

(d) Is the Klein bottle homeomorphic to the connected sum of two projective planes? Justify your answer. **(2 marks)**

4 (i) (a) State the inclusion/exclusion principle for the Euler characteristic. **(3 marks)**

(b) Calculate the Euler characteristic of the cube

$$C \ := \ \{(x,y,z) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\}.$$

(8 marks)

(ii) (a) Calculate the Euler characteristic of the connected sum of a torus and a Klein bottle. **(4 marks)**

(b) Express the connected sum of a torus and a Klein bottle in standard form. **(4 marks)**

(iii) (a) Let S be a convex polyhedron formed from squares and regular hexagons. Show that if three faces meet at each vertex then S has exactly six square faces. **(5 marks)**

(b) Explicitly describe a convex polyhedron with exactly two hexagonal faces. **(1 mark)**

End of Question Paper