The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2017–2018**

**MAS346 Groups and Symmetry** **2 hours 30 minutes**

*Attempt all the questions. The allocation of marks is shown in brackets (note that Question 3 is out of 15 marks).*

***Additional Material:*** *Diagram for Question 2*

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**Do not remove it from the hall**

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to be completed by student

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**1** (i) (a) Define an isometry of $\mathbf{R}^2$. Let $\mathrm{Isom}_2$ be the set of isometries of $\mathbf{R}^2$. Show that it is closed under composition and taking inverses.

***(7 marks)***

(b) Let $f, g \in \mathrm{Isom}_2$ be given by $f(x) = Ax + a$ and $g(x) = Bx + b$, where $A, B \in O_2$ and $a, b \in \mathbf{R}^2$. Find $D$ and $d$ such that $fgf^{-1}(x) = Dx + d$ for all $x$.

***(4 marks)***

(ii) Define the homomorphism $\psi : \mathrm{Isom}_2 \to O_2$ and show that it is a homomorphism.

***(3 marks)***

(iii) Let $H \leq \mathrm{Isom}_2$ be a subgroup, and let $a$ be a point in $\mathbf{R}^2$. Put

$$\sigma_a(H) = \{A \in O_2 | T_a A T_a^{-1} \in H\}.$$

(a) Prove that $\sigma_a(H) \subseteq \psi(H)$.

***(3 marks)***

(b) Let $g \in \mathrm{Isom}_2$ such that $g(a) = a$. Show that if $f := T_{-a} g T_a$ then $f(x) = Ax$ for some $A \in O_2$.

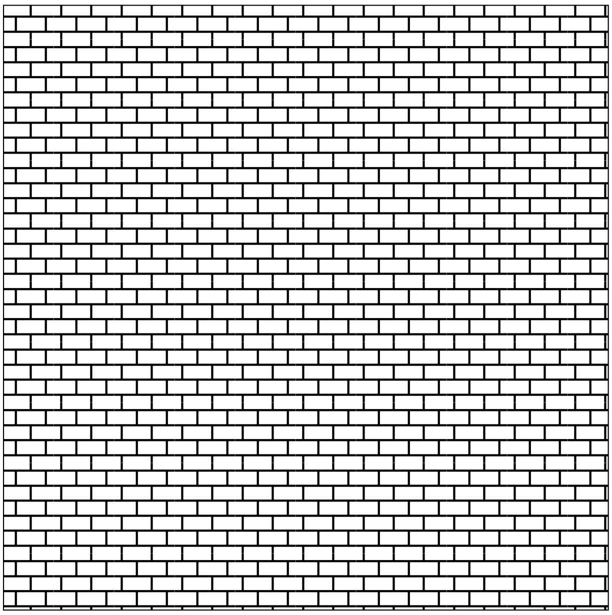
***(2 marks)***

(c) Prove that if $\mathrm{orb}_H(a) = \mathrm{orb}_{\mathrm{Trans}(H)}(a)$ then $\sigma_a(H) = \psi(H)$.
[Hint: Given $h \in H$ first find a translation that composed with $h$ fixes $a$. Then apply (b) to obtain an element of $\sigma_a(H)$. Now use properties of $\psi$ to prove that this element equals $\psi(h)$.]

***(6 marks)***

**2** (i) For any subgroup $H \leq \mathrm{Isom}_2$ recall that $\psi(H) \leq O_2$ is its point group and $\mathrm{Trans}(H) \leq \mathbf{R}^2$ its translation subgroup. Explain which properties $\psi(H)$ and $\mathrm{Trans}(H)$ need to satisfy for $H$ to be a wallpaper group. ***(3 marks)***

(ii) Let $G$ be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(a) Describe geometrically *all* the translations, reflections and rotations (if any) in $G$. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. Specify one more element of $G$ that is not a translation, rotation or reflection. ***(8 marks)***

(b) Find a list of four isometries that generate $G$. Justify your answer. ***(9 marks)***

(c) Find a rotation center that is not on any reflection line. Write this rotation in terms of the generators you found in (b). ***(5 marks)***

**3** (i) (a) Give the definition of an action of a group $G$ on a set $X$. ***(3 marks)***

(b) Given a group action explain how to define the corresponding map $\phi : G \to S(X)$ and prove that it is a homomorphism taking values in $S(X)$. ***(5 marks)***

(ii) Let a group $G$ act on itself by $g * x := gxg^{-1}$.

(a) Show that this defines a group action. ***(2 marks)***

(b) Determine the kernel of the corresponding homomorphism $\phi : G \to S(G)$. ***(3 marks)***

(c) Prove that for every $g \in G$ the map $\phi(g) \in S(G)$ defines a homomorphism from $G$ to $G$. ***(2 marks)***

**4** (i) (a) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(b) Determine the number of Sylow 2- and 3-subgroups of $S_4$. ***(6 marks)***

(ii) (a) Give the definition of a simple group. ***(2 marks)***

(b) Prove that there is no simple group of order 12. ***(5 marks)***

(iii) A finite group $G$ with order divisible by a prime $p$ acts on the set of Sylow $p$-subgroups by $g * P = gPg^{-1}$. [You do not need to prove this.]

(a) Apply the Sylow theorems to express the order of the stabilizer of a Sylow $p$-subgroup under this action in terms of $|G|$ and the number $n_p$ of Sylow $p$-subgroups of $G$. ***(3 marks)***

(b) Use the action of $G$ on the cosets of such a stabilizer to show that if $|G|$ does not divide $(n_p)!$ then $G$ is not simple. ***(4 marks)***

## End of Question Paper

# Diagram for Question 2

**Your registration number:** ________________