The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

MAS350 Measure and Probability **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Let S be a set. Give precise definitions of

(a) A σ–*algebra* Σ of subsets of S. ***(3 marks)***

(b) A *measure* m on the measurable space (S, Σ). ***(3 marks)***

What does it mean for m to be *finite*, and what is the *total mass* of a finite measure m? Briefly show how a probability measure may be obtained from such a finite measure. ***(3 marks)***

(ii) Let $m_1, m_2, \ldots, m_N$ be measures on the measurable space (S, Σ) and $c_1, c_2, \ldots, c_N$ be non–negative real numbers. Show that $m = \sum_{i=1}^{N} c_i m_i$ is a measure on (S, Σ). ***(4 marks)***

If m_i is a probability measure for $i = 1, 2, \ldots, N$, what is the total mass of m? State a condition on the c_i's which ensures that m is a probability measure. ***(2 marks)***

(iii) Let Σ be a σ–algebra of subsets of S.

(a) If $A, B \in \Sigma$, show that $A \cap B \in \Sigma$. ***(1 mark)***

(b) If $A, B, C \in \Sigma$, is it true that

$$(A \cup B) \cap C - (A \cap B) \cup C \in \Sigma?$$

Give careful reasoning to support your conclusion. ***(2 marks)***

(iv) We construct a variant on the Cantor set as follows. Start with the interval $[0, 1]$ and remove the middle $1/5$ to obtain the set D_1. Then remove the middle $1/5$ of each of the disjoint intervals comprising D_1 to obtain D_2. Iterate this procedure to obtain a sequence of sets (D_n), and define $D = \bigcap_{n=1}^{\infty} D_n$. Deduce a formula for the Lebesgue measure of D_n (there is no need to formally prove this) and hence obtain the Lebesgue measure of D, stating clearly any results you use to justify this last deduction. ***(7 marks)***

2 Throughout this question (S, Σ) is a measurable space, and $\mathbb{R}$ is equipped with its usual Borel σ–algebra $\mathcal{B}(\mathbb{R})$.

(i) Give two (distinct) equivalent formulations of what it means for $f : S \to \mathbb{R}$ to be measurable. ***(2 marks)***

(ii) Let f and g be measurable functions from S to $\mathbb{R}$.

(a) Prove that the set $\{f > g\} \in \Sigma$, where

$$\{f > g\} := \{x \in S; f(x) > g(x)\}.$$

(4 marks)

(b) Use (a) to show that $f - g$ is measurable. ***(3 marks)***

[Hint: You may use the fact that $g + a$ is measurable for all $a \in \mathbb{R}$, where $(g + a)(x) = g(x) + a$, for all $x \in \mathbb{R}$.]

(c) Deduce that $\{f = g\} \in \Sigma$, where

$$\{f = g\} := \{x \in S; f(x) = g(x)\}.$$

(4 marks)

[Hint: You may use the fact that $\{a\} \subseteq \mathcal{B}(\mathbb{R})$ for all $a \in \mathbb{R}$.]

(d) A *fixed point* of f is a solution of the equation $f(x) = x$. Show that the set of all fixed points of f is measurable. ***(2 marks)***

(iii) What can you say about the measurability of the set

$$E := \{x \in \mathbb{R}; \sqrt{1 + x^2}\sin(x + 3) = e^x\}?$$

Give brief arguments to support your conclusions. ***(2 marks)***

(iv) Let (A_n) be an increasing sequence of sets in Σ and define the associated indicator functions in the usual way:

$$\mathbf{1}_{A_n}(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

for all $n \in \mathbb{N}$.

(a) Show that $\lim_{n\to\infty} \mathbf{1}_{A_n}(x)$ exists for all $x \in S$, and that the limit is a measurable function. ***(3 marks)***

(b) Construct a sequence (B_n) of mutually disjoint sets in Σ so that for all $n \in \mathbb{N}$,

$$\mathbf{1}_{A_n} = \sum_{r=1}^{n} \mathbf{1}_{B_r},$$

and use this to prove that $\mathbf{1}_A(x) = \lim_{n\to\infty} \mathbf{1}_{A_n}(x)$ for all $x \in S$, where $A = \bigcup_{n\in\mathbb{N}} A_n$. ***(5 marks)***

3 Throughout this question (S, Σ, m) is a fixed measure space.

(i) (a) Write down the general form of a non–negative *simple function*, explaining carefully the properties of any numbers and sets that appear in your expression. ***(3 marks)***

(b) Give a formula for the *Lebesgue integral* of the function in (a). ***(2 marks)***

(ii) Consider the function $g : \mathbb{R} \to \mathbb{R}$ that is given as follows:

$$g(x) = \begin{cases} -2 & \text{if } -5 \leq x < -3 \\ 1, & \text{if } -3 \leq x < -2, \\ 6 & \text{if } -2 \leq x < 1, \\ 4 & \text{if } 1 \leq x < 3, \\ -3 & \text{if } 3 \leq x < 5, \\ 0, & \text{if } |x| \geq 5 \end{cases}$$

(a) Write g explicitly as a simple function. ***(2 marks)***

(b) Define $h(x) = g(x) + 3$ for all $x \in \mathbb{R}$. Is h a simple function? If so write it explicitly. ***(3 marks)***

(c) Calculate $\displaystyle\int_{[-10,10]} g(x)dx$, $\displaystyle\int_{[-15,15]} h(x)dx$ and $\displaystyle\int_{\mathbb{R}} |g(x)|dx$. ***(8 marks)***

(iii) Let $f : S \to \mathbb{R}$ be an integrable function

(a) Prove that

$$\left| \int_S f(x)dm(x) \right| \leq \int_S |f(x)|dm(x),$$

making sure that you carefully introduce any tools that you need for the proof. ***(4 marks)***

(b) Explain why $\sin \circ f$ is a measurable function from S to $\mathbb{R}$, and deduce that

$$\left| \int_S \sin(f(x))dm(x) \right| < \infty.$$

(3 marks)

[Hint: You may use the inequality $|\sin(y)| \leq |y|$ for all $y \in \mathbb{R}$.]

4 Throughout this question (S, Σ, m) is a fixed measure space.

(i) State *Fatou's lemma* ***(2 marks)***

(ii) Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ which converges pointwise to a function f. Suppose that we are given an integrable function $g : S \to [0, \infty)$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$.

(a) Show that f_n is integrable for all $n \in \mathbb{N}$. ***(2 marks)***

(b) Show that f is integrable [Hint: Use Fatou's lemma]. ***(3 marks)***

(c) Briefly explain why Fatou's lemma may be applied to the sequence $(g + f_n)$, and hence show that

$$\int_S f dm \leq \liminf_{n\to\infty} \int_S f_n dm.$$

(4 marks)

(d) Repeat the argument of (c), with $g + f_n$ replaced with $g - f_n$ to show that

$$\limsup_{n\to\infty} \int_S f_n dm \leq \int_S f dm.$$

(4 marks)

(e) Complete the proof of *Lebesgue's dominated convergence theorem* by deducing that

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

(3 marks)

(iii) Let $a > 1$. Explain why $\sum_{n=1}^{\infty} \int_1^a e^{-nx} dx$ exists, and hence show that

$$\lim_{a\to\infty} \sum_{n=1}^{\infty} \int_1^a e^{-nx} dx = 1 - \log_e(e - 1).$$

(7 marks)

5 (i) (a) Let (S, Σ, m) be a measure space, and $f : S \to \mathbb{R}$ be a measurable function. Suppose that $|f|^n$ is integrable for some $n \in \mathbb{N}$. Prove the following generalisation of Markov's inequality:

$$m\{x \in S; |f(x)| > a\} \leq \frac{1}{a^n} \int_S |f|^n dm,$$

where $a > 0$. ***(4 marks)***

(b) Reformulate the result of (a) for a random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$, and having a finite nth moment. ***(2 marks)***

(c) We are given two random variables Y_1 and Y_2 defined on $(\Omega, \mathcal{F}, P)$. We are told that $\mathbb{E}(|Y_1|) = 12$ and $\mathbb{E}(|Y_2|^5) = 3$. We want to estimate $P(|Y_1| > 4)$ and $P(|Y_2| > 4)$ as accurately as possible. Is the inequality of (b) of any value in either of these cases? Present evidence, in the form of explicit calculations, to support your conclusions. ***(4 marks)***

(ii) Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Explain what it means for a sequence (X_n) of random variable to converge to a random variable X

(I) in mean square, (II) in probability, (III) almost surely.

(3 marks)

(a) Show that convergence in mean square implies convergence in probability. ***(2 marks)***

(b) State (without proof) any results that directly relate almost sure convergence to either of the other two types. ***(3 marks)***

(c) Let X be a fixed random variable for which $\mathbb{E}(|X|^2) < \infty$. Let (A_n) be a sequence of events in $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} A_n = \Omega$. Show that $(X\mathbf{1}_{A_n})$ converges in probability to a limit as $n \to \infty$, and find this limit. ***(7 marks)***

[Hint: $\lim_{n \to \infty} \mathbf{1}_{A_n}(\omega) = 1$, for all $\omega \in \Omega$.]

End of Question Paper