Data provided: Supplementary Formula Sheet

The University Of Sheffield.

MAS352

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Stochastic Processes and Financial Mathematics **3 hours**

Candidates should attempt ***ALL*** *questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 Let $\Omega = \{1,2,3,4\}$ and consider the set

$$\mathcal{A} = \{\emptyset, \Omega, \{1\}, \{2,3\}\}$$

(a) Show that $\mathcal{A}$ is not a σ-field. ***(2 marks)***

(b) List the elements of $\sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$. ***(5 marks)***

(c) Define $X : \Omega \to \mathbb{R}$ by $X(\omega) = \omega$. Is the function X $\sigma(\mathcal{A})$-measurable? Justify your answer. ***(2 marks)***

2 Let X be a random variable. Explain why

$$Y = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

is a random variable. ***(4 marks)***

You may use standard results about measurability of sums, products and limits of random variables, providing they are clearly stated.

3 Let X_1 be a random variable with distribution $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = \frac{1}{2}$. Define a sequence of random variables (X_n) by specifying that, for all $n \in \mathbb{N}$,

$$X_{n+1} = -X_n.$$

Let $Y = X_1$.

Which of the following statements are true and which are false? Briefly justify your answer in each case.

(a) The sequence (X_n) is constant.

(b) The sequence (X_n) is deterministic.

(c) The sequence (X_n) is bounded.

(d) $X_n \stackrel{d}{\to} Y$ as $n \to \infty$.

(e) $X_n \stackrel{\mathbb{P}}{\to} Y$ as $n \to \infty$.

(10 marks)

4 This questions concerns the binomial model, in discrete time, with two assets, cash and stock.

A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.

(a) Let $h_t = (x_t, y_t)$, for $t = 1, \ldots, T$, be a portfolio strategy, and let V_t^h denote its value process.

(i) What does it mean for (h_t) to be self-financing? ***(2 marks)***

(ii) What does it mean for (h_t) to replicate the contingent claim $\Phi(S_T)$? ***(1 mark)***

(iii) State a condition on the parameters d, r and u which is equivalent to the binomial model being free of arbitrage. ***(2 marks)***

(b) Take $T = 2$, and let the parameters of the model be $p_u = p_d = 0.5$, $u = 1.5$, $d = 0.75$, $r = 0$ and $s = 80$. Consider the contingent claim

$$\Phi(S_T) = \max(S_T - 50)$$

with exercise time $T = 2$.

Draw a recombining tree of the stock price process at times $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for $\Phi(S_T)$, at each node, along with a portfolio strategy that replicates $\Phi(S_T)$. ***(10 marks)***

5 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, with the distribution of X_n given by

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = \frac{1}{2n^2}, \qquad \mathbb{P}[X_n = 0] = 1 - \frac{1}{n^2}.$$

Define the random walk $S_n = \sum_{i=1}^{n} X_i$, where $S_0 = 0$.

(a) Show that S_n is a martingale, with respect the filtration $\mathcal{F}_n$ generated by X_n. ***(5 marks)***

(b) Calculate $\mathbb{E}[|X_n|]$ and hence show that

$$\mathbb{E}[|S_n|] \le \sum_{i=1}^{n} \frac{1}{i^2}$$

for all n. ***(3 marks)***

(c) Deduce that there exists a random variable S_∞ such that $S_n \stackrel{a.s.}{\to} S_\infty$ as $n \to \infty$.

Explain briefly why this means that, almost surely, the random walk S_n will not make infinitely many jumps. ***(4 marks)***

6 (a) State the definition of Brownian motion. ***(5 marks)***

(b) Let B_t be a Brownian motion, let $\mathcal{F}_t$ be its generated filtration, and let $0 \leq s \leq t$.

Use the definition of Brownian motion to show that $\mathbb{E}[B_t \,|\, \mathcal{F}_s] = B_s$ and that $\mathbb{E}[B_t^2 - B_s^2 \,|\, \mathcal{F}_s] = t - s$. ***(6 marks)***

(c) State one difference, and one similarity, between Brownian motion and random walks. ***(2 marks)***

7 Let B_t be a standard Brownian motion and let $Z_t = (\sin B_t)^2$.

(a) Show that Z_t is a solution of the stochastic differential equation

$$dZ_t = (1 - 2Z_t)\, dt + 2 \sin B_t \cos B_t \, dB_t.$$

(5 marks)

(b) Show that $f(t) = \mathbb{E}[Z_t]$ satisfies the ordinary differential equation

$$f'(t) + 2f(t) = 1$$

Express $f(t)$ as an explicit function of t and hence show that $f(t) \to \frac{1}{2}$ as $t \to \infty$. ***(6 marks)***

(c) Is the stochastic process $C_t = Z_t - \mathbb{E}[Z_t]$ a Brownian motion? Justify your answer. ***(3 marks)***

8 Within the Black-Scholes model, in the risk-neutral world $\mathbb{Q}$ the stock price satisfies the stochastic differential equation

$$dS_t = rS_t \, dt + \sigma S_t \, dB_t.$$

Show that, in the risk neutral world $\mathbb{Q}$, the stochastic process $M_t = e^{-rt} S_t$ is a martingale. ***(5 marks)***

A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.

9 (a) Let B_t be a standard Brownian motion. Let $\alpha \in \mathbb{R}$ and $\sigma > 0$, and let X_t be an Ito process satisfying $X_0 > 0$ and

$$dX_t = \alpha X_t \, dt + \sigma X_t \, dB_t.$$

Find the stochastic differential dZ_t where $Z_t = \log X_t$. ***(5 marks)***

(b) Let $\Phi(S_T) = \log(S_T)$ be a contingent claim, at time $T > 0$. Within the Black-Scholes model, show that the value of this contingent claim at time $t \in [0, T]$, is given by

$$e^{-r(T-t)} \left(\log S_t + (r - \tfrac{1}{2}\sigma^2)(T-t) \right).$$

(7 marks)

(c) (i) Consider a portfolio with value $V(t, S_t)$ at time t. State what it means for this portfolio to be delta neutral at time t. ***(1 mark)***

(ii) Suggest one reason why it might be advantageous to hold a delta neutral portfolio. ***(1 mark)***

(iii) Let $r = 1, S_0 = 1$ and $\sigma^2 = 2$. Suppose that, at time 0, our portfolio consists of a contract with contingent claim $\Phi(S_T) = \log(S_T)$.

Using the pricing formula from (b), find the amount x of stock that we would need to additionally include into our portfolio, in order to make our portfolio delta neutral at time 0. ***(4 marks)***

End of Question Paper

MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- $X_n \stackrel{d}{\to} X \Leftrightarrow$ for any $x \in \mathbb{R}$, $\lim_{n\to\infty} \mathbb{P}[X_n \le x] \to \mathbb{P}[X \le x]$.
- $X_n \stackrel{\mathbb{P}}{\to} X \Leftrightarrow$ for any $a > 0$, $\lim_{n\to\infty} \mathbb{P}[|X_n - X| > a] = 0$.
- $X_n \stackrel{a.s.}{\to} X \Leftrightarrow \mathbb{P}[X_n \to X \text{ as } n \to \infty] = 1$.
- $X_n \stackrel{L^p}{\to} X \Leftrightarrow \mathbb{E}[|X_n - X|^p] \to 0$ as $n \to \infty$.

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants r (discrete interest rate), p_u and p_d (probabilities of stock price increase/decrease), u and d (factors of stock price increase/decrease), and s (initial stock price).

The value of x in cash, held at time t, will become $x(1+r)$ at time $t+1$.

The value of a unit of stock S_t, at time t, satisfies $S_{t+1} = Z_n S_n$, where $\mathbb{P}[Z_t = u] = p_u$ and $\mathbb{P}[Z_t = d] = p_d$, with initial value $S_0 = s$.

When $d < 1 + r < u$, the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \qquad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time $t = 0, 1, 2, \dots, T$. The case $T = 1$ is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale M_n and a stopping time T, holds if any one of the following conditions is fulfilled:

(a) T is bounded.

(b) M_n is bounded and $\mathbb{P}[T < \infty] = 1$.

(c) $\mathbb{E}[T] < \infty$ and there exists $c \in \mathbb{R}$ such that $|M_n - M_{n-1}| \le c$ for all n.

MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t,x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t)\, dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Feyman-Kac formula

Suppose that $F(t,x)$, for $t \in [0,T]$ and $x \in \mathbb{R}$, satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \alpha(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\beta(t,x)^2 \frac{\partial^2 F}{\partial x^2}(t,x) &= 0 \\ F(T,x) &= \Phi(x). \end{aligned}$$

Then $F(t,x) = \mathbb{E}_{t,x}\left[\Phi(X_T)\right]$, where X_u satisfies $dX_u = \alpha(u, X_u)\,dt + \beta(u, X_u)\,dB_u$.

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t\,dt$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$, with initial value S_0.

At time $t \in [0, T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0,$$
$$F(T,s) = \Phi(s).$$

The unique solution F satisfies

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t]$$

for all $t \in [0, T]$. Here, the 'risk-neutral world' $\mathbb{Q}$ is the probability measure under which S_t satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0, 1]$, we define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability η_j the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.