RESTRICTED OPEN BOOK EXAMINATION
Data provided: Table of χ^2 quantiles

MAS371

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Applied Probability **2 hours**

Restricted Open Book Examination.
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator which conforms to University regulations.
Your should attempt ***all three*** *questions. Total marks 60.*
You may use standard results from the lecture notes without derivation, provided you state them clearly.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) The general discrete-time Markov chain with two states has transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

where p and q are unknown parameters. A sequence of observations of the state of the chain $X_0, X_1, X_2, \ldots, X_n$ is made. Write down the log-likelihood $l(P|X_0, \ldots, X_n)$ in terms of N_{ij}, the number of observed transitions from state i to state j, and hence find the observed information $J(P|X_0, \ldots, X_n)$.
(4 marks)

(ii) A particle moves between the vertices of a triangle, numbered clockwise as (1,2,3), according to a discrete-time Markov chain. At each time step, it moves one place anti-clockwise with probability p, moves one place clockwise with probability q, or remains where it is with probability $1-p-q$, where $0 < p, q < 1/2$.

(a) Write down the transition probability matrix for this model.
(1 mark)

(b) If a sequence of observations $X_0, \ldots, X_n$ of the process is made, show that the likelihood for the parameters p and q can written as

$$L(p, q|X_0, \ldots, X_n) \propto p^{N_A} q^{N_C} (1-p-q)^{N_R}$$

where N_A, N_C, N_R are summaries of the data which you should define.
(3 marks)

(c) Derive the observed information matrix for p and q. ***(6 marks)***

(d) If the sequence observed is

3, 2, 2, 3, 2, 3, 1, 1, 1, 3, 2, 1, 1, 3, 3,

obtain the maximum likelihood estimates for p and q. ***(4 marks)***

(e) Explain how to calculate approximate standard errors for p and q. (You do **not** need to carry out this calculation.) ***(2 marks)***

2 Mice can be classified genetically into one of three types, labelled 0, 1 and 2. Breeding the mice in the laboratory, and observing the genetic type of one particular mouse, and then its first offspring, and so on, gives a sequence of linked observations $X_0, X_1, \ldots, X_n$ say, each taking values in $S = \{0, 1, 2\}$.

(i) A possible model for how the observations are linked is a discrete-time Markov chain with transition probability matrix

$$\begin{pmatrix} p_{00} & p_{01} & 0 \\ p_{10} & p_{11} & p_{12} \\ 0 & p_{21} & p_{22} \end{pmatrix}$$

where the 0s denote values that are forced to be zero by the basic laws of genetics, and the other entries are all positive.

In one particular sequence of the form described above, the observations are as follows:

2, 2, 1, 1, 0, 1, 1, 1, 2, 1, 2, 1, 0, 1, 0, 0, 0, 0, 1, 1.

(a) Calculate the maximum likelihood estimates of the elements of the transition matrix for the model described above and the estimated standard error for p_{11}. ***(4 marks)***

(b) Calculate an approximate 95% confidence interval for $p_{10} - p_{12}$. ***(4 marks)***

(ii) A particular genetic model suggests that the transition matrix in (i) should be

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Write down an expression for the likelihood for this model, based on an observation of a sequence of genetic types summarised by $n_{ij}, i, j \in \{0, 1, 2\}$, where n_{ij} denotes the number of individuals of type i in the sequence whose first offspring were of type j. (You may assume that $n_{02} = n_{20} = 0$, so that the observed sequence is *possible* under this model.) ***(2 marks)***

(iii) (a) Test the hypothesis that the data in (i) come from the model in (ii), against the alternative of the more general form in (i). ***(8 marks)***

(b) Comment briefly on the relationship between your results in (iii)(a) and in (i)(b). ***(2 marks)***

3 (i) The number of females in a captive population of a rare species is monitored over 6 months. (The number of males is ignored, but is always positive.) The initial size of the population is 10. The times (in months, since the start of monitoring) at which births occurred are 0.7, 1.9, 2.5, 4.0, 5.1, 5.9. (These are births of females; any births of males are ignored.) Similarly, the times at which deaths occurred are 1.3, 3.7, 5.5.

Assuming that the population can be modelled by a linear birth-death process, calculate estimates and standard errors for the birth and death rates per individual per month. ***(7 marks)***

(ii) A k-server ($k \geq 2$) queuing process $X(t)$ has an arrival rate that depends on whether the system is empty, full or in between, so that

$$\lambda_i = \begin{cases} \alpha & i = 0 \\ \beta & 1 \leq i \leq k-1 \\ \gamma & i \geq k \end{cases}$$

with $\alpha > \beta > \gamma$. The process has a completion rate per server of μ.

(a) Write down an expression for the probability $\Pr(X(t + \delta t) = 1 | X(t) = 1)$ for small but finite δt. ***(2 marks)***

(b) State the equations that must be satisfied by a stationary distribution for this process. Hence give a condition for such a distribution to exist. What can be said about its form? ***(6 marks)***

(c) Write down the log-likelihood for α, β, γ based on complete observation of the system over an interval $[0, t]$. In the case $k = 2$, derive an expression for the maximum likelihood estimator of the parameter β. ***(5 marks)***

End of Question Paper

Table of the qth quantile of the χ^2 distribution with ν degrees of freedom, $\chi^2_{q,\nu}$

		ν								
		1	2	3	4	5	6	7	8	9
	0.10	0.016	0.211	0.584	1.064	1.610	2.204	2.833	3.490	4.168
	0.50	0.455	1.386	2.366	3.357	4.351	5.348	6.346	7.344	8.343
q	0.90	2.706	4.605	6.251	7.779	9.236	10.645	12.017	13.362	14.684
	0.95	3.841	5.991	7.815	9.488	11.070	12.592	14.067	15.507	16.919
	0.99	6.635	9.210	11.345	13.277	15.086	16.812	18.475	20.090	21.666
		ν								
		10	20	30	40	50	60	70	80	90
	0.10	4.87	12.44	20.60	29.05	37.69	46.46	55.33	64.28	73.29
	0.50	9.34	19.34	29.34	39.34	49.33	59.33	69.33	79.33	89.33
q	0.90	15.99	28.41	40.26	51.81	63.17	74.40	85.53	96.58	107.57
	0.95	18.31	31.41	43.77	55.76	67.50	79.08	90.53	101.88	113.15
	0.99	23.21	37.57	50.89	63.69	76.15	88.38	100.43	112.33	124.12