The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Time Series **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

RESTRICTED OPEN BOOK EXAMINATION

Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.

There are 60 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Consider that 100 observations of a time series $\{y_t\}$ gave values of the sample partial autocorrelation function (PACF) and sample autocorrelation function (ACF) tabulated below:

Lag h	1	2	3	4	5
PACF ($a_h^{(h)}$)	0.72	0.41	0.14	0.07	0.02
ACF (r_h)	$\star$	$\star\star$	0.70	0.62	0.44

(a) Find the values of $\star$ and $\star\star$. ***(4 marks)***

(b) Test whether $\{y_t\}$ is consistent with moving average models: MA(1) and MA(2). ***(4 marks)***

(c) Test whether $\{y_t\}$ is consistent with autoregressive models: AR(1), AR(2) and AR(3). ***(2 marks)***

(d) Based on your analysis above, propose an overall model that you feel is expected to fit the data well. ***(1 mark)***

(ii) Consider the autoregressive time series model

$$y_t = \frac{1}{4}y_{t-1} + \epsilon_t - \frac{1}{2}\epsilon_{t-1} + \frac{1}{3}\epsilon_{t-2}, \tag{1}$$

where ϵ_t is white noise with variance 1.

(a) Show that model (1) is causal and invertible. ***(3 marks)***

(b) Calculate the variance of y_t. ***(6 marks)***

2 Suppose that observations $y_1, y_2, \ldots, y_n$ are generated from the autoregressive (AR) model of order two:

$$y_t = \alpha y_{t-1} - (1-\alpha) y_{t-2} + \epsilon_t,$$

where α is a parameter and ϵ_t is a Gaussian white noise with variance σ^2.

(i) Write down the conditional likelihood function $L(\alpha, \sigma^2; y_{1:n})$ and the conditional log-likelihood function $\ell(\alpha, \sigma^2; y_{1:n})$ of the parameters α and σ^2, based on observation $y_{1:n} = \{y_1, y_2, \ldots, y_n\}$. ***(3 marks)***

(ii) Using (i) and adopting conditional least squares, show that the likelihood estimates of α and σ^2 satisfy

$$\hat{\alpha} = \frac{\sum_{t=3}^n y_t y_{t-1} + \sum_{t=3}^n y_t y_{t-2} + \sum_{t=3}^n y_{t-2}^2 + \sum_{t=3}^n y_{t-1} y_{t-2}}{\sum_{t=3}^n y_{t-1}^2 + \sum_{t=3}^n y_{t-2}^2 + 2\sum_{t=3}^n y_{t-1} y_{t-2}}$$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{t=3}^n \left[y_t - \hat{\alpha} y_{t-1} + (1-\hat{\alpha}) y_{t-2} \right]^2$$

(11 marks)

(iii) (a) If $y_1 = 1$, $y_2 = 2$, $y_3 = 0$ and $y_4 = 1$, calculate the maximum likelihood estimate $\hat{\alpha}$ of α. ***(2 marks)***

(b) Based, on the observed data above, calculate the 2-step ahead forecast of the observation y_6. ***(3 marks)***

(c) If y_6 is observed to be equal to $y_6 = 0.1$, calculate the corresponding two-step ahead forecast error of the forecast in part (b) above. ***(1 mark)***

3 Consider that a time series $\{y_t\}$ is generated from an ARIMA(1,1,1) model, so that

$$y_t - y_{t-1} = \alpha(y_{t-1} - y_{t-2}) + \epsilon_t + \gamma\epsilon_{t-1},$$

where α is the AR parameter, γ is the MA parameter and $\{\epsilon_t\}$ is a Gaussian white noise sequence with variance equal to 1.

Define the state vector

$$\beta_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \end{bmatrix}.$$

(i) Write down a state space representation for y_t, i.e. express y_t as a state space model:

$$y_t = x^\top \beta_t + \delta_t,$$
$$\beta_t = F\beta_{t-1} + \zeta_t.$$

In your answer you should:

(a) specify the components x, F, δ_t and ζ_t; *(2 marks)*

(b) write down the distributions of δ_t and ζ_t. *(2 marks)*

(ii) Suppose that at time $t = 2$, the posterior distribution of β_2 is

$$\beta_2 \mid y_{1:2} \sim N\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

where $y_{1:2} = \{y_1 = 1, y_2 = 0\}$ denotes the information available at time $t = 2$.

(a) If $y_3 = 1$, perform a step of the Kalman filter and hence derive the posterior distribution of

$$\beta_3 \mid y_{1:3},$$

where $y_{1:3} = \{y_1 = 1, y_2 = 0, y_3 = 1\}$. *(14 marks)*

(b) Given information $y_{1:3}$, find the posterior distribution of ϵ_3. *(2 marks)*

End of Question Paper