The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017-2018

Analytical Dynamics and Classical Field Theory — **3 Hours**

Answer all questions.

1 (i) Write down the Euler–Lagrange equation for the Lagrangian $L = L(q, \dot{q}, t)$, describing the dynamics of a particle with mass m moving in one dimension with position q and velocity $\dot{q} = dq/dt$ under the influence of a potential $V(q)$. Define all quantities appearing in the equation. ***(5 marks)***

(ii) Assume that the Lagrange function L does not explicitly depend on time, i.e. $L = L(q, \dot{q})$. Show that the following equation holds:

$$L - \dot{q}\frac{\partial L}{\partial \dot{q}} = \text{constant}.$$

Hint: Calculate dL/dt. ***(8 marks)***

(iii) Use the equation above to show that $E = \dot{q}^2/2 + V(q)$ is conserved. What is the interpretation of E? ***(7 marks)***

2 (i) Consider a system with N degrees of freedom, described by a Lagrange–function $L(q_i, \dot{q}_i, t)$ (with $i = 1, ..., N$). Define the canonical momenta P_i and define the Hamilton–function H. L depends on q_i, $\dot{q}_i$ and time t. What does H depend on?

(6 marks)

(ii) Write down Hamilton's equations. ***(3 marks)***

(iii) Show that if the Hamilton–function does not depend explicitly on time, it is constant. What is the interpretation of H in this case? ***(3 marks)***

(iv) Consider two particles with masses m_1 and m_2, respectively. They are moving in one dimension under the influence of a potential $V(q_1, q_2) = \kappa_1 q_1^2 + \kappa_2 q_2^2 + \lambda q_1 q_2$, where q_1 is the (generalised) coordinate describing particle 1, q_2 describing particle 2 and κ_1, κ_2 and λ are constants. Find the Hamilton–function from first principles (i.e. do **not** assume that $H = T + V$, where T is the kinetic energy and V is the potential energy) and write down Hamilton's equations. Show that the equations of motion for q_1 and q_2 can be written as

$$\begin{aligned}
\ddot{q}_1 + 2\frac{\kappa_1}{m_1} q_1 + \frac{\lambda}{m_1} q_2 &= 0 \, , \\
\ddot{q}_2 + 2\frac{\kappa_2}{m_2} q_2 + \frac{\lambda}{m_2} q_1 &= 0 \, .
\end{aligned}$$

(8 marks)

3 Consider the Lagrangian of a complex scalar field $\Phi(x)$, given by

$$\mathcal{L} = \eta^{\mu\nu}(\partial_\mu \Phi^*)(\partial_\nu \Phi) - V(\Phi\Phi^*),$$

where V is an arbitrary function depending only on the product $\Phi\Phi^*$.

(i) Show that the Lagrangian above is invariant under the transformation $\Phi \to \Phi' = e^{i\theta}\Phi$, where θ is constant. ***(4 marks)***

(ii) Show that the equations of motion for Φ^* and Φ are given by

$$\begin{aligned} \partial^\mu \partial_\mu \Phi^* &= -\frac{\partial V}{\partial \Phi}\,, \\ \partial^\mu \partial_\mu \Phi &= -\frac{\partial V}{\partial \Phi^*}\,. \end{aligned}$$

Hint: Treat Φ and Φ^* as independent variables. ***(6 marks)***

(iii) State Noether's theorem and argue that, because of part (i), there must be a conserved current. You don't have to write down the equation for the current. ***(3 marks)***

(iv) You are given that the Noether current is

$$J_\mu = \Phi^*(\partial_\mu \Phi) - (\partial_\mu \Phi^*)\Phi.$$

Show that $\partial^\mu J_\mu = 0$, provided that Φ and Φ^* satisfy the equations of motion given above. ***(7 marks)***

4 For this question, you are given that the Christoffel–symbols are given by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right).$$

(i) Consider the following line–element:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2(\sin^2\theta \; d\phi^2 + d\theta^2),$$

where the coordinates are $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and $\alpha(r)$ and $\beta(r)$ are undetermined functions of r. Write down all non–vanishing components of the metric tensor $g_{\mu\nu}$. Write down all non–vanishing components of the inverse metric $g^{\mu\nu}$ as well. ***(6 marks)***

(ii) Show that

$$\begin{aligned} \Gamma^{r}_{tt} &= e^{2(\alpha-\beta)} \frac{\partial \alpha(r)}{\partial r} \\ \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^{r}_{\phi\phi} &= -re^{-2\beta}\sin^2\theta \\ \Gamma^{t}_{tr} &= \frac{\partial \alpha}{\partial r}. \end{aligned}$$

(7 marks)

(iii) You are given the following two components of the Ricci–tensor:

$$\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)} \left[\frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2 - \frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial \alpha}{\partial r} \right] , \\ R_{rr} &= -\frac{\partial^2 \alpha}{\partial r^2} - \left(\frac{\partial \alpha}{\partial r}\right)^2 + \frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial \beta}{\partial r} . \end{aligned}$$

Assume that the energy–momentum tensor of matter vanishes (locally). What does that imply for R_{tt} and R_{rr}? Furthermore, show that in this case

$$\alpha(r) = -\beta(r) + C \ ,$$

where C is a constant. ***(7 marks)***

5 For this question, you are given that the covariant derivative of a four–vector V_α is $V_{\alpha;\beta} = \partial_\beta V_\alpha - \Gamma^\mu_{\alpha\beta} V_\mu$. You are also given that under the coordinate transformation $x^\mu \to x'^\mu$ the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \ . \qquad (1)$$

(i) Show that if $V_{\alpha;\beta} = 0$ holds in one coordinate system, it holds in all coordinate systems. [Hint: no explicit calculation is needed for this question! What is the transformation rule for the covariant derivative under coordinate transformations?]

(2 marks)

(ii) Consider an infinitesimal coordinate transformation of the form $x'^\mu = x^\mu + \epsilon v^\mu(x)$, with $\epsilon \ll 1$ and v^μ is a vector–field.

(a) Show that, to first order in ϵ

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \epsilon \frac{\partial v^\mu}{\partial x^\nu} \ . \qquad (2)$$

(4 marks)

(b) Use equations (1) and (2) to show that, to first order in ϵ

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \epsilon \left(g_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} + g_{\mu\beta} \frac{\partial v^\beta}{\partial x^\nu} \right) \ . \qquad (3)$$

(6 marks)

(c) Show that if $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$, the vector field v^α obeys

$$v^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} + g_{\mu\beta} \frac{\partial v^\beta}{\partial x^\nu} = 0 \ .$$

Hint: You need to use eq. (3) and a Taylor expansion of $g'_{\mu\nu}(x')$ for x' close to x.

(8 marks)

End of Question Paper