The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017–2018

MAS422 Magnetohydrodynamics — 2 hours

Answer all four questions. Formulae are on the last page.

1 (i) Using the solenoidal condition, show that the following magnetic field (in cylindrical polar coordinates, (r, ϕ, z)) has no sources or sinks of flux and determine its current

$$\boldsymbol{B}(r, \phi, z) = B_0 \left(r, r^4 e^{-r^2}, -2z \right).$$

(6 marks)

(ii) Consider the magnetic induction equation in the case where the magnetic diffusivity $\eta = 0$. Use the standard vector identity

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

together with Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

to show that the induction equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{u} \cdot \nabla)\frac{\mathbf{B}}{\rho} = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{u}.$$

(8 marks)

(iii) (a) Write down the steady state induction equation, including the magnetic diffusion term for uniform diffusivity. ***(2 marks)***

(b) Consider an incompressible flow

$$\mathbf{v} = (V \sin(kx)\cos(kz), 0, -V\cos(kx)\sin(kz))$$

and a magnetic field of the form $\mathbf{B} = (0, 0, B_z(x))$.
Using the equation obtained in (a), find the first order ordinary differential equation (ODE) for B_z. ***(6 marks)***

1 (continued)

(c) Solve the ODE obtained in (b) for B_z using the boundary condition that, at $x = 0$, $\dfrac{dB_z}{dx} = 0$. ***(3 marks)***

2 (i) Consider a magnetic field with components

$$B_x(x, z) = \frac{\partial A}{\partial z}, \qquad B_y = B_y(x, z), \qquad B_z(x, z) = -\frac{\partial A}{\partial x}$$

where $A = A(x, z)$.

(a) Show that $\nabla \cdot \mathbf{B} = 0$. ***(2 marks)***

(b) Show that $\mathbf{B} \cdot \nabla A = 0$ and that projections of field lines in the xz-plane are given by A=constant. ***(5 marks)***

(c) Show that, if the Lorentz force

$$\mathbf{J} \times \mathbf{B} = \mathbf{0}$$

then

$$B_y = B_y(A)$$

(7 marks)

and A satisfies

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} + B_y \frac{dB_y}{dA} = 0.$$

(3 marks)

(ii) Sketch the field lines of the magnetic field whose components are

$$B_x = -y, \qquad B_y = 5x.$$

(8 marks)

3 (i) If a plasma is incompressible and the radius of a flux tube is decreased by a factor 3, use conservation of mass and flux to determine what happens to its length and field strength? ***(8 marks)***

(ii) Given a velocity field $\mathbf{u} = (yz, -xz, 0)$ and the initial magnetic field $\mathbf{B}(\mathbf{x}, 0) = (x, -y, 0)$, find $\mathbf{B}(\mathbf{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\mathbf{u}$ and applying the Cauchy solution. ***(17 marks)***

4 (i) Consider a magnetic field in cylindrical polar coordinates (r, ϕ, z). If a bounded magnetic field $\mathbf{B} = \mathbf{B}(r)$ varies with r alone, why can it not possess a radial component (B_r)? ***(3 marks)***

(ii) The equilibrium that requires total pressure balance has the gradient of the sum of the gas pressure and magnetic pressure as zero, i.e.

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0,$$

with μ_0 as the vacuum permeability.

If $\mathbf{B} = B_0 \tan h(\frac{x}{L})\hat{\mathbf{y}}$,
what is the gas pressure p?
what is the current density $\mathbf{J}$? ***(4 marks)***

(iii) Consider the full set of ideal MHD equations as given below:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} = -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p(\nabla \cdot \mathbf{u}) = \mathbf{0}$$

where ρ, $\mathbf{u}$, p and $\mathbf{B}$ denote the density, velocity, pressure and magnetic field respectively. The ratio of the specific heat is denoted by γ.

(a) Now, assume the equilibrium quantities to be at rest and constant (i.e. we do not consider inhomogeneous plasma). Write down the linearised MHD equations using the above equations for small perturbations. Show, clearly, the MHD variables expressed (i.e. equilibrium and perturbed quantities). ***(10 marks)***

(b) Use the linearised equations obtained in (a) and derive the governing equation for the perturbed velocity, $\mathbf{u_1}$. Use the definition of sound speed and Alfvén speed. ***(3 marks)***

(c) By assuming a plane wave solution of the form $\mathbf{u_1} = \hat{u_1} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ in Cartesian coordinates system, find the equation for $\mathbf{u_1}$ in terms of ω and $\mathbf{k}$. ***(2 marks)***

4 (continued)

(d) If $\mathbf{B} = B_0\hat{\mathbf{z}}$ and $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are unit vectors along x and z directions respectively, write down the three equations for the three components of $\mathbf{u_1}$ using the equation obtained in (c). ***(3 marks)***

Formulae Sheet

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla\cdot\mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla\times\mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$

vector identity:

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

End of Question Paper