The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2017-2018**

Advanced Operations Research **2 hours**

Attempt all FOUR questions.

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Do not remove it from the hall

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1 (i) You are given the following linear programming problem (LPP):

$$\max \quad z = 3x_1 - x_2$$

subject to $x_1, x_2 \geq 0$ and

$$2x_1 + x_2 \leq 6, \qquad x_1 - x_2 \leq 1.$$

(a) Find the optimal solution for the problem with the simplex method. Clearly state the optimal solution. ***(10 marks)***

(b) Sketch the feasible region of the LPP, and mark the vertices corresponding to the basic solutions in each tableau. ***(4 marks)***

(ii) The two-phase method is used to solve the LPP:

$$\max\ z = 3x_1 - 5x_2$$

subject to constraints $x_1, x_2 \geq 0$ and

$$x_1 + 2x_2 \leq 7, \quad 4x_1 + 3x_2 = 6, \quad 2x_1 - x_2 \geq 1.$$

Find the initial tableau in phase 1 and process it so that it is suitable to be solved by the simplex method. **Do NOT proceed further.** ***(3 marks)***

(iii) A company assembles four types of cars in three factories. The table below lists the types of cars that can be assembled in each factory.

Factory	Car types
1	1,2,4
2	2
3	1,3,4

The daily production capacities of the three factories are 25, 18 and 30 cars, respectively. The daily demands for the four types of cars are no more than 20, 17, 35 and 12 units, respectively.

The company wants to use a maximal flow model to find the plan that will maximise the profit without over-production. Sketch and annotate properly the network corresponding to the maximal flow model. **Do NOT try to find the mixed integer linear programming formulation or the numerical solution**. ***(8 marks)***

2 A manufacturer makes two types of products, $P1$ and $P2$. Both are made from the same raw material. The required labour and raw material for a unit of the products are given in the table below. Also shown in the table are the selling prices for the products.

	Raw material (kg/unit)	labour (hour/unit)	Price (£/unit)
$P1$	5	2	25
$P2$	7	6	45

Raw material can be purchased from two providers, provider A and B. The following information is known:

- If A is chosen as the regular provider, he can offer a price of £3/kg for the first 200 kilograms, and £2/kg afterwards. However, A can supply only 380 kilograms of the material per week. If more is needed, the manufacturer will have to purchase it from B at a cost of £4/kg.
- If B is chosen as the regular provider, he can offer a price of £4/kg for the first 100 kilograms, and £2/kg afterwards. B can supply only 400 kilograms of the material per week. If more is needed, the manufacturer will have to purchase it from A at a cost of £4/kg.
- The manufacturer will need to choose one and only one of the two as his regular provider.
- The manufacturer has 480 hours of free labour available each week.

Derive a single mixed integer-linear programming model for the problem whose solution provides the optimal weekly production schedule to maximise the manufacturer's profit. **Hint: when you define the cost function, you need to take into account the expense on raw material**. **Find the formulation only; do NOT proceed further.** ***(25 marks)***

3 You are given the following primal linear programming problem

$$\min\ z = c^T x, \quad \text{subject to} \quad A_1 x \leq b_1,\ A_2 x \geq b_2, \text{ and } x \geq 0. \tag{1}$$

(i) Define its Lagrangian function and show that its Lagrangian dual problem can be written as

$$\max\ v = b_2^T y_2 - b_1^T y_1, \tag{2}$$

subject to

$$A_2^T y_2 - A_1^T y_1 \leq c,\ y_1 \geq 0, \text{ and } y_2 \geq 0. \tag{3}$$

(12 marks)

(ii) Define the shadow costs of the primal constraints and find their values in terms of the solutions of the dual problem. ***(4 marks)***

(iii) Show that the dual problem of the problem defined by Equations (2) and (3) is the same as the primal problem in Equation (1). ***(9 marks)***

4 A car maker assembles three different models of cars on two assembly lines. The models are called F1, F2, F3, and the two assembly lines are line A and line B. All cars go through both assembly lines. Relevant information is given in the table below.

	Processing time (hour)			Yearly capacity (hour)
	F1	F2	F3	
line A	4	10	6	5300
line B	6	8	12	5400
Unit profit (k£)	6	12	10	

To determine the production schedule that maximises the total profit, we define x_1, x_2 and x_3 as the numbers of units of F1, F2, and F3 to be produced, respectively, and formulate the following model:

$$\max \ z = 6x_1 + 12x_2 + 10x_3 \qquad \text{(in k£)}$$

subject to $x_1, x_2, x_3 \geq 0$, and

$$4x_1 + 10x_2 + 6x_3 \leq 5300,$$
$$6x_1 + 8x_2 + 12x_3 \leq 5400.$$

As a first approximation, we solve the problem as a linear programming problem, i.e. we allow non-integer solutions. Introducing slack variables x_4 and x_5 for the first and the second constraints, respectively, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	2/7	6/7	3/7	48000/7
x_2	0	1	-3/7	3/14	-1/7	2550/7
x_1	1	0	18/7	-2/7	5/14	2900/7

(i) Find the optimal cost from the optimal tableau, as well as the optimal solution for the primal variables, and the optimal solution for the dual variables. ***(3 marks)***

(ii) Find the optimality range for the cost coefficient of x_1, and give a brief practical interpretation. ***(6 marks)***

(iii) Suppose that the capacity for assembly line B is reduced to 3000 hours per year due to machine breakdown. Is the original optimal solution still feasible? If not, use the dual simplex method to find the new optimal feasible solution. ***(10 marks)***

(iv) A change in the design of model F2 has reduced its processing time to 8 hours and 4 hours on assembly line A and B, respectively. Other data in the model are not changed. Does the optimal basis remain the same? Support your answer with necessary calculation. ***(6 marks)***

End of Question Paper