Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1 (i) (a) What is a topological space? If $X$ and $Y$ are topological spaces define the product topology on $X \times Y$. (5 marks)

(b) Suppose that $T, X$ and $Y$ are topological spaces and $f : T \rightarrow X \times Y$. Writing $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ for the projections, show that if the composites $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous, so too is $f$. (4 marks)

(ii) (a) If $X$ is a topological space and $a, b \in X$ what is a path from $a$ to $b$? (2 marks)

(b) Suppose that $\omega$ is a path from $a$ to $b$ and $\sigma$ is a path from $b$ to $c$ define the concatenated path $\omega \cdot \sigma$.

Consider the relation $a \sim b$ if there is a path from $a$ to $b$. Show that this is an equivalence relation. (We let $\pi_0(X)$ denote the set of equivalence classes). (5 marks)

(c) If $f : X \rightarrow Y$ is a continuous function, explain how to define the induced map $f_* : \pi_0(X) \rightarrow \pi_0(Y)$. (3 marks)

(d) Show that the map $f : \pi_0(X \times Y) \rightarrow \pi_0(X) \times \pi_0(Y)$ induced by the projections is a bijection. (6 marks)
2 (i) (a) What is a covering map? (4 marks)

(b) State the Path Lifting Lemma for a covering map \( p : Y \to X \), and explain how it can be used to define a function

\[
\ell : \pi_1(X, x_0) \to p^{-1}(x_0),
\]

where \( x_0 \in X \) (you need not check that your definition is independent of the choices you make). State conditions under which \( \ell \) is a bijection. (8 marks)

(ii) (a) Consider the torus \( T = S^1 \times S^1 \) and the self-map \( f : T \to T \) defined by \( f(w, z) = (-w, \bar{z}) \) (where \( w, z \) are complex numbers of modulus 1) and notice that \( f^2 \) is the identity. Take the quotient space \( K = T/\sim \) where \( (w, z) \sim f(w, z) \) and give it the quotient topology. Show that the quotient map \( p : T \to K \) is a covering map. (4 marks)

(b) Choose basepoints \( \tilde{x}_0 = (1, 1) \) and \( x_0 = p(\tilde{x}_0) \). Show that \( p_* : \pi_1(T, \tilde{x}_0) \to \pi_1(K, x_0) \) is injective. Let \( \tilde{\sigma} \) be any path from \( f(\tilde{x}_0) \) to \( \tilde{x}_0 \) and let \( h \in \pi_1(K, x_0) \) be the class of the loop \( \sigma := p \circ \tilde{\sigma} \). Show that for any element \( g \in \pi_1(K, x_0) \), either \( g \) or \( gh \) is in the image of \( p_* \). Deduce that \( \pi_1(K, x_0) \) is generated by three elements. (9 marks)

3 (i) What is a chain complex of abelian groups? What is the homology of such a chain complex? (5 marks)

(ii) Show that if \( K \) is a \( n \)-dimensional simplicial complex, then \( H_n(K) \) is a free abelian group. Show that if \( L \) is a subcomplex of \( K \) which includes all simplices of dimension \( \leq d \) then \( H_i(L) = H_i(K) \) for \( i \leq d - 1 \). (7 marks)

(iii) Let \( \Delta^n \) be the standard \( n \)-simplex with vertex set \( \{ e_0, e_1, \ldots, e_n \} \). Write down \( H_*(\Delta^n) \).

Let \( (\Delta^n)^{(d)} \) be the simplicial complex of all faces of dimension \( \leq d \).

Draw pictures of \( (\Delta^3)^{(k)} \) for \( k = 0, 1, 2 \). (4 marks)

Show that \( H_i((\Delta^n)^{(d)}) = 0 \) unless \( i = 0 \) or \( i = d \). For \( n \geq 3 \), calculate the homology of \( (\Delta^n)^{(n-1)} \) and \( (\Delta^n)^{(n-2)} \). (9 marks)
4 (i) State the Mayer-Vietoris Theorem for calculating the homology of a simplicial complex $K = L \cup M$ expressed as the union of two subcomplexes $L$ and $M$. (5 marks)

(ii) Let $X$ be formed by sticking a Möbius strip to a 2-torus $T^2$ by identifying the boundary circle with some circle in $T^2$. Suppose $X$ may be triangulated using a simplicial complex $K = L \cup M$ with $L$ being a triangulation of the 2-torus $T^2$, and let $M$ being a triangulation of the Möbius strip.

Write down $H_*(L), H_*(M), H_*(L \cap M)$ and identify the map induced by the inclusion $L \cap M \to M$, making any assumptions about the triangulations that are convenient. (8 marks)

Write down the Mayer-Vietoris long exact sequence for $K = L \cup M$, and identify $H_0(K), H_2(K)$. Identify two possibilities for $H_1(K)$ and show that they both occur. (12 marks)

5 Are the following true or false. Justify your answers.

(i) Any continuous self-map of the closed unit ball $B^3$ in $\mathbb{R}^3$ has a fixed point. (5 marks)

(ii) Writing $d(P, Q)$ for the Euclidean distance from $P$ to $Q$, the space $X := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (n, 0)) < 1/2 \text{ for some } n \in \mathbb{Z}\}$ is homeomorphic to $Y := \{(x, y) \in \mathbb{R}^2 \mid (x, y) \notin \mathbb{Z} \times \{0\}\}$. (5 marks)

(iii) There is a covering map $K^2 \to T^2$ from the Klein bottle to the torus. (5 marks)

(iv) One may remove a finite number of points from $\mathbb{R}^2$ and obtain a space homotopy equivalent to the projective plane $\mathbb{R}P^2$. (5 marks)

(v) The space $X$ obtained by deleting the $z$ axis from $\mathbb{R}^3$ is homotopy equivalent to a 1-dimensional complex. (5 marks)

End of Question Paper