



The  
University  
Of  
Sheffield.

MAS436

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2017–2018

MAS436: Functional Analysis

2 hours 30 minutes

Answer **four** questions. If you answer more than four questions, only your best four will be counted.

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers,  $\mathbb{R}$ , or the field of complex numbers,  $\mathbb{C}$ .

1 (i) Let  $C_b(\mathbb{R})$  be the complex vector space of all bounded continuous functions  $f: \mathbb{R} \rightarrow \mathbb{C}$ . Prove that we have a norm on  $C_b(\mathbb{R})$  defined by the formula

$$\|f\| = \sup\{|f(t)| \mid t \in \mathbb{R}\}.$$

(6 marks)

(ii) State the definition of a *Banach space* and a *closed subspace* of a normed vector space. Prove that any closed subspace of a Banach space is complete. (6 marks)

(iii) Prove that the space  $C_b(\mathbb{R})$  is a Banach space. (9 marks)

(iv) Let  $C_0(\mathbb{R})$  be the subspace of  $C_b(\mathbb{R})$  consisting of functions  $f \in C_b(\mathbb{R})$  such that  $f(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ . Prove that  $C_0(\mathbb{R})$  is closed. (4 marks)

2 (i) Let  $V$  and  $W$  be normed vector spaces. Say what is meant by the statement that a linear map  $T: V \rightarrow W$  is *bounded*, and prove that if  $T$  is a bounded linear map, then the kernel,  $\ker T$ , is closed. **(5 marks)**

(ii) Let  $C^\infty[0, 1]$  be the vector space of all real-valued infinitely-differentiable functions  $f: [0, 1] \rightarrow \mathbb{R}$ , equipped with the norm

$$\|f\| = \sup\{|f(t)| \mid t \in [0, 1]\}.$$

Let  $C_0^\infty[0, 1]$  be the subspace of all infinitely differentiable functions  $f: [0, 1] \rightarrow \mathbb{R}$  such that  $f(0) = 0$ .

Prove that we have a bijective bounded linear map  $T: C^\infty[0, 1] \rightarrow C_0^\infty[0, 1]$  defined by the formula

$$(Tf)(x) = \int_0^x f(t) dt \quad x \in [0, 1].$$

**(6 marks)**

(iii) Does the map  $T$  have a *continuous* inverse? Justify your answer.

**(4 marks)**

(iv) Say what is meant by an open map, and state the open mapping theorem.

**(3 marks)**

(v) Prove that  $C_0^\infty[0, 1]$  is a closed subspace of the space  $C^\infty[0, 1]$ . Is the normed vector space  $C^\infty[0, 1]$  complete? Justify your answer. **(7 marks)**

- 3 (i) State the Stone-Weierstrass theorem for complex-valued functions. Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let  $f_k(z) = z^k$ . Show that the span of the set  $\{z^k \mid k \in \mathbb{Z}\}$  is dense in  $C(\mathbb{T})$ .

(7 marks)

- (ii) Let  $C_\rho[0, 2\pi]$  be the Banach space of continuous functions  $f: [0, 2\pi] \rightarrow \mathbb{C}$  such that  $f(0) = f(2\pi)$ , under the norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 2\pi]\}.$$

Write down an isometric isomorphism  $\alpha: C(\mathbb{T}) \rightarrow C_\rho[0, 2\pi]$ . (3 marks)

- (iii) Let  $e_k(t) = \exp(ikt)$ . Show that  $\{e_k \mid k \in \mathbb{Z}\}$  is an orthonormal basis for the space  $L^2[0, 2\pi]$ . Your answer should include a proof that a dense subset of  $C_\rho[0, 2\pi]$  under the above norm is also a dense subset of  $L^2[0, 2\pi]$  under its usual norm. (9 marks)

- (iv) Let  $f(x) = x$ . Find coefficients  $a_k \in \mathbb{C}$  such that the series

$$\sum_{k=-\infty}^{\infty} a_k e_k = f$$

in the space  $L^2[0, 2\pi]$ .

Use the above to evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(6 marks)

4 (i) Let  $f, g \in L^1(\mathbb{R})$  be continuous functions. Show that we have a well-defined function  $f * g \in L^1(\mathbb{R})$  defined by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

(6 marks)

(ii) For continuous functions  $f, g \in L^1(\mathbb{R})$ , show that, when taking the Fourier transform,  $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$ . (6 marks)

(iii) Let  $\alpha > 0$ . Let  $f \in L^1(\mathbb{R})$  be continuous. Let  $g(x) = f(\alpha x)$ . Show that, when considering the Fourier transform, we have  $\hat{g}(\omega) = \frac{1}{\alpha} \hat{f}\left(\frac{\omega}{\alpha}\right)$ . (3 marks)

(iv) Let  $\sigma > 0$ . Let

$$g_\sigma = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

Calculate the Fourier transform  $\hat{g}_\sigma$ . (6 marks)

(v) Show that for any  $\sigma, \tau > 0$ , we have  $g_\sigma * g_\tau = g_{\sqrt{\sigma^2 + \tau^2}}$ . (4 marks)

You may assume here, if desired, the Fourier inversion theorem.

5 (i) Let  $V$  be a Banach space, and let  $T: V \rightarrow V$  be a bounded linear map. Say what is meant by an *eigenvalue* of  $T$  and the *spectrum* of  $T$ . Prove that every eigenvalue of  $T$  lies in the spectrum. (6 marks)

(ii) Let  $V$  be a complex Banach space. Let  $D$  be an open subset of  $\mathbb{C}$ . What is meant by saying a function  $f: D \rightarrow V$  is *holomorphic*? State *Liouville's theorem* for holomorphic functions with values in  $V$ . (4 marks)

(iii) Let  $T: V \rightarrow V$  be a bounded linear map, and let  $D = \mathbb{C} \setminus \text{Spectrum}(T)$ . Let  $R(\lambda) = (T - \lambda I)^{-1}$ . Prove that  $R(\lambda) - R(\mu) = (\lambda - \mu)R_\lambda R_\mu$  for all  $\lambda, \mu \in D$ , and that  $R: D \rightarrow \text{Spectrum}(T)$  is holomorphic. You may use the Carl Neumann criterion without proof. (8 marks)

(iv) Use the above to prove that the spectrum of  $T$  is always non-empty. Give an example of a bounded linear map with no eigenvalues. (7 marks)

End of Question Paper