The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018**

Optics and Symplectic Geometry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper I *denotes an identity matrix and* J *denotes a matrix of the form* $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. *The standard symplectic form* Ω *on* $\mathbb{R}^{2n}$ *is defined by* $\Omega(Z,Z') = Q \cdot P' - P \cdot Q'$, *where* $Z = (Q,P)$ *and* $Z' = (Q',P')$ *are elements of* $\mathbb{R}^{2n}$.

Throughout the paper you may use standard results of linear algebra without proof, provided that you state the results which you use clearly.

1 Consider the set $\widetilde{\mathscr{L}_3}$ of all oriented lines in $\mathbb{R}^3$ and the manifold $K_3 \subseteq \mathbb{R}^6$ defined by

$$K_3 = \{(X,Y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |X| = 1,\ X \cdot Y = 0\}.$$

(a) Describe, using neat diagrams as needed, the maps $\widetilde{\mathscr{L}_3} \to K_3$ and $K_3 \to \widetilde{\mathscr{L}_3}$ which establish a bijection between $\widetilde{\mathscr{L}_3}$ and K_3. (You are not asked to prove that these maps give a bijection.) **(8 marks)**

(b) Denote by Φ_L the map $\widetilde{\mathscr{L}_3} \to \widetilde{\mathscr{L}_3}$ which sends each oriented line to the same line with the reverse orientation. Give a formula for the map $\Phi_K\colon\ K_3 \to K_3$ that corresponds to Φ_L.

Write U for the subset of K_3 which corresponds, under the above bijection, to the set of oriented lines that are tangent to the unit sphere. Determine U. **(6 marks)**

(c) Prove that K_3 is a manifold, and determine the tangent space $T_{(X,Y)}(K_3)$ at each $(X,Y) \in K_3$. State clearly any general theorem about smooth functions that you use. You are not asked to prove that the function(s) you use are smooth. **(11 marks)**

2 (i) Define what it means for a $2n \times 2n$ matrix S to be symplectic. $\mathbf{(2\ marks)}$

(ii) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

(a) Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold.

(b) Assume that S is symplectic and invertible. Give a formula for S^{-1} in block form. $\mathbf{(6\ marks)}$

(iii) (a) Take $S \in Sp(4)$ and define a map $\Theta\colon\ (\mathbb{R}^4)^4 \to \mathbb{R}$ by

$$\begin{aligned} &\Theta(Z_1, Z_2, Z_3, Z_4) \\ &= \Omega(Z_1, Z_2)\Omega(Z_3, Z_4) - \Omega(Z_1, Z_3)\Omega(Z_2, Z_4) + \Omega(Z_1, Z_4)\Omega(Z_2, Z_3), \end{aligned}$$

where $Z_1, \ldots, Z_4 \in \mathbb{R}^4$. You may assume that Θ is multilinear. Show that it is skew-symmetric.

Using the following result, prove that $\det S = +1$.

> *Let $\Theta\colon\ (\mathbb{R}^4)^4 \to \mathbb{R}$ be a multilinear, skew-symmetric form such that $\Theta(E_1, E_2, E_3, E_4) = +1$, where E_1, E_2, E_3, E_4 is the standard basis of $\mathbb{R}^4$. Then $\Theta(Z_1, Z_2, Z_3, Z_4)$ is the determinant of the matrix with columns Z_1, Z_2, Z_3, Z_4, for any $Z_1, Z_2, Z_3, Z_4 \in \mathbb{R}^4$.*

(b) Define a map $\Theta\colon\ (\mathbb{R}^6)^6 \to \mathbb{R}$ which may be expected to be multlilinear, skew-symmetric, and have $\Theta(E_1, E_2, E_3, E_4, E_5, E_6) \neq 0$, where $E_1, E_2, E_3, E_4, E_5, E_6$ is the standard basis of $\mathbb{R}^6$. You are not asked to prove any of these properties. $\mathbf{(17\ marks)}$

3 (i) (a) Let $S(t)$ be a symplectic matrix for all $t \in \mathbb{R}$ with $S(0) = I_{2n}$, the identity matrix.

Let H denote $\left.\frac{d}{dt}S(t)\right|_{t=0}$, the derivative of $S(t)$ at $t = 0$.

Prove that

$$H^{\mathsf{T}}J + JH = 0. \tag{1}$$

You may assume, without proof, that differentiation of matrix–valued functions obeys the usual rules for sums and products, with due attention to the order of factors in products.

(b) Now let H be a $2n \times 2n$ matrix which satisfies (1) and is such that $I - H$ is invertible. Show that

$$S = (I + H)(I - H)^{-1}$$

is a symplectic matrix. **(15 marks)**

(ii) Let S be a $2n \times 2n$ matrix.

Denote the first n columns of S by $F_1, \ldots, F_n$ and the last n columns by $G_1, \ldots, G_n$. Prove *any two* of the following equations:

$$\begin{cases} \Omega(F_i, F_j) = 0 & \text{for all } 1 \leqslant i, j \leqslant n, \\ \Omega(G_i, G_j) = 0 & \text{for all } 1 \leqslant i, j \leqslant n, \\ \Omega(F_i, G_j) = 0 & \text{for all } 1 \leqslant i, j \leqslant n,\ i \neq j, \\ \Omega(F_i, G_i) = 1 & \text{for all } 1 \leqslant i \leqslant n. \end{cases}$$

You may, if you wish, proceed by seperating each F_i and G_j into pairs of elements of $\mathbb{R}^n$. **(10 marks)**

4 (i) In Gaussian optics, refraction across a parabolic lens corresponds to a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ -m & 1 \end{bmatrix}$$

where $m \in \mathbb{R}$. Also in Gaussian optics, change of reference vertical corresponds to a matrix of the form

$$\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$$

where we refer to $w > 0$ as the optical distance.

(a) Calculate the matrix N corresponding to refraction through two parabolic lenses with coefficents m_1 and m_2, separated by optical distance $w > 0$.

(b) Show that every symplectic matrix

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $b \neq 0$ may be written as a matrix N as in (a) for unique values of m_1, m_2 and $w > 0$. **(6 marks)**

(ii) Let H_3 denote the Heisenberg group, consisting of all matrices of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$. Denote by $\mathfrak{h}_3$ the vector space of all matrices of the form

$$\begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$.

Define the adjoint and coadjoint actions of H_3 and calculate the orbits of each action. **(19 marks)**

End of Question Paper