The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Galois Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper K denotes a subfield of $\mathbb{C}$ which contains $\mathbb{Q}$.
All field extensions are finite.

1 (i) Find the roots of the cubic equation $x^3 - 3x + 4 = 0$. Simplify your answers as much as possible, making use of ω, a primitive cube root of 1, as needed. Identify the real root.

You may, if you wish, set $x = u + v$ in terms of new variables u and v, and proceed by obtaining equations for $u^3 + v^3$ and uv. **(13 marks)**

(ii) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Now let p and q be distinct primes. Show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.

Find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $\mathbb{Q}(\sqrt{p} + \sqrt{q})$ is the splitting field of $f(x)$. List the roots of $f(x)$. **(12 marks)**

2 (i) In each of (a), (b), (c), find the roots of the polynomial, and using them, determine the degree over $\mathbb{Q}$ of the splitting fields.

You may use, if you wish, a shifted Eisenstein criterion.

(a) $x^4 + 1$,

(b) $x^6 + 1$,

(c) $x^6 + x^3 + 1$.

(15 marks)

(ii) Let G be a subgroup of a symmetric group S_n, where $n \geqslant 3$. Define what it means for G to be *transitive*.

Prove that if n is prime and G is transitive and contains a transposition, then $G = S_n$. **(10 marks)**

3 (i) Let $\xi = e^{2\pi i/11}$, and put

$$\beta = \xi + \frac{1}{\xi} = 2\cos\left(\frac{2\pi}{11}\right).$$

(a) Show that β satisfies a quintic equation over $\mathbb{Q}$, and write it down.

(b) Write $\gamma = \xi + \xi^3 + \xi^4 + \xi^5 + \xi^9$. Expand γ^2 in powers of ξ, and hence deduce that $\gamma^2 + \gamma + 3 = 0$. Show that $\mathbb{Q}(\sqrt{-11}) \subseteq \mathbb{Q}(\xi)$.

(c) State the general result(s) about the Galois groups of cyclotomic extensions from which it follows that $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is cyclic with 10 elements.

(d) Choose a generator θ of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, being sure to define θ explicitly. In terms of θ, write down the subgroups of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$.

(15 marks)

(ii) Let ξ be a primitive nth root of unity, where $n \geqslant 3$, and write $\beta = \xi + \dfrac{1}{\xi}$.

(a) Show that ξ satisfies a quadratic equation over $\mathbb{Q}(\beta)$ and deduce that $[\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] \leqslant 2$.

(b) Show that $\mathbb{Q}(\beta) \subseteq \mathbb{R}$, and deduce that $\xi \notin \mathbb{Q}(\beta)$.

(c) Deduce that $[\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] = 2$ and hence calculate $[\mathbb{Q}(\beta) : \mathbb{Q}]$, expressing your answer in terms of the φ function, defined so that $\varphi(k) = |U(\mathbb{Z}_k)|$ for $k \geqslant 2$.

(10 marks)

4 (i) State in full the Galois correspondence. Your answer should include the relation between indexes of subgroups and degrees of field extensions, and the significance of normal subgroups. **(6 marks)**

(ii) Define what it means for a group G to be soluble. **(3 marks)**

(iii) Let G be the dihedral group with 8 elements. Write the elements of G as

$$I, R, R^2, R^3, F, RF, R^2F, R^3F,$$

where R has order 4 and F has order 2, and $FRF = R^{-1}$.

(a) Give the subgroup lattice of G and *deduce from it* that G is soluble. **(12 marks)**

(b) List the subgroups in the subgroup lattice that are *not* normal, giving brief justifications for your answers. **(4 marks)**

End of Question Paper