



The
University
Of
Sheffield.

MAS452/MAS6052

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2017–2018**

Stochastic Processes and Financial Mathematics

3 hours

*Candidates should attempt **ALL** questions.*

The maximum marks for the various parts of the questions are indicated.

The paper will be marked out of 100.

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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1 Let $\Omega = \{1, 2, 3, 4\}$ and consider the set

$$\mathcal{A} = \{\emptyset, \Omega, \{1\}, \{2, 3\}\}$$

- (a) Show that \mathcal{A} is not a σ -field. (2 marks)
- (b) List the elements of $\sigma(\mathcal{A})$, the σ -field generated by \mathcal{A} . (5 marks)
- (c) Define $X : \Omega \rightarrow \mathbb{R}$ by $X(\omega) = \omega$. Is the function X $\sigma(\mathcal{A})$ -measurable? Justify your answer. (2 marks)

2 Let X be a random variable. Explain why

$$Y = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

is a random variable. (4 marks)

You may use standard results about measurability of sums, products and limits of random variables, providing they are clearly stated.

3 Let X_1 be a random variable with distribution $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = \frac{1}{2}$. Define a sequence of random variables (X_n) by specifying that, for all $n \in \mathbb{N}$,

$$X_{n+1} = -X_n.$$

Let $Y = X_1$.

Which of the following statements are true and which are false? Briefly justify your answer in each case.

- (a) $X_n \xrightarrow{d} Y$ as $n \rightarrow \infty$.
- (b) $X_n \xrightarrow{\mathbb{P}} Y$ as $n \rightarrow \infty$.
- (c) $X_n \xrightarrow{a.s.} Y$ as $n \rightarrow \infty$.

(6 marks)

- 4 This questions concerns the binomial model, in discrete time, with two assets, cash and stock.

A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.

- (a) Let $h_t = (x_t, y_t)$, for $t = 1, \dots, T$, be a portfolio strategy, and let V_t^h denote its value process.
- (i) What does it mean for (h_t) to replicate the contingent claim $\Phi(S_T)$? **(1 mark)**
- (ii) State a condition on the parameters d, r and u which is equivalent to the binomial model being free of arbitrage. **(2 marks)**
- (b) Take $T = 2$, and let the parameters of the model be $p_u = p_d = 0.5$, $u = 1.5$, $d = 0.75$, $r = 0$ and $s = 2$. Consider the contingent claim

$$\Phi(S_T) = \max(S_T - 10)$$

with exercise time $T = 2$.

Draw a recombining tree of the stock price process at times $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for $\Phi(S_T)$, at each node, along with a portfolio strategy that replicates $\Phi(S_T)$. **(10 marks)**

- 5 Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables, with the distribution of X_n given by

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = \frac{1}{2n^2}, \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{n^2}.$$

Define the random walk $S_n = \sum_{i=1}^n X_i$, where $S_0 = 0$.

- (a) Show that S_n is a martingale, with respect the filtration \mathcal{F}_n generated by X_n . **(5 marks)**
- (b) Calculate $\mathbb{E}[|X_n|]$ and hence show that

$$\mathbb{E}[|S_n|] \leq \sum_{i=1}^n \frac{1}{i^2}$$

for all n . **(3 marks)**

- (c) Deduce that there exists a random variable S_∞ such that $S_n \xrightarrow{a.s.} S_\infty$ as $n \rightarrow \infty$.

Explain briefly why this means that, almost surely, the random walk S_n will not make infinitely many jumps. **(4 marks)**

- (d) Let

$$T = \inf\{n \geq 0 ; \text{for all } m \geq n, S_m = S_\infty\}.$$

Is T a stopping time? Give brief justification for your answer. **(2 marks)**

- (e) Show that $\mathbb{E}[S_T] = \mathbb{E}[S_\infty] = 0$. **(5 marks)**

6 (a) State the definition of Brownian motion. *(5 marks)*

(b) Let B_t be a Brownian motion, let \mathcal{F}_t be its generated filtration, and let $0 \leq s \leq t$.

Use the definition of Brownian motion to show that $\mathbb{E}[B_t | \mathcal{F}_s] = B_s$ and that $\mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s] = t - s$. *(6 marks)*

(c) State one difference, and one similarity, between Brownian motion and random walks. *(2 marks)*

7 Let B_t be a standard Brownian motion and let $Z_t = (\sin B_t)^2$.

(a) Show that Z_t is a solution of the stochastic differential equation

$$dZ_t = (1 - 2Z_t) dt + 2 \sin B_t \cos B_t dB_t.$$

(5 marks)

(b) Show that $f(t) = \mathbb{E}[Z_t]$ satisfies the ordinary differential equation

$$f'(t) + 2f(t) = 1$$

Express $f(t)$ as an explicit function of t and hence show that $f(t) \rightarrow \frac{1}{2}$ as $t \rightarrow \infty$. *(6 marks)*

- 8 (a) Let B_t be a standard Brownian motion. Let $\alpha \in \mathbb{R}$ and $\sigma > 0$, and let X_t be an Ito process satisfying $X_0 > 0$ and

$$dX_t = \alpha X_t dt + \sigma X_t dB_t.$$

Find the stochastic differential dZ_t where $Z_t = \log X_t$. **(5 marks)**

- (b) Let $\Phi(S_T) = \log(S_T)$ be a contingent claim, at time $T > 0$. Within the Black-Scholes model, show that the value of this contingent claim at time $t \in [0, T]$, is given by

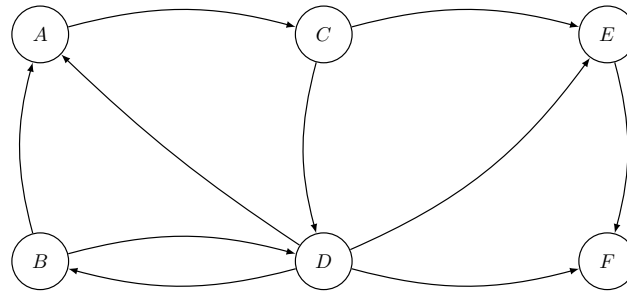
$$e^{-r(T-t)} \left(\log S_t + \left(r - \frac{1}{2}\sigma^2 \right) (T - t) \right).$$

(7 marks)

A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.

- (c) (i) Consider a portfolio with value $V(t, S_t)$ at time t . State what it means for this portfolio to be delta neutral at time t . **(1 mark)**
- (ii) Suggest one reason why it might be advantageous to hold a delta neutral portfolio. **(1 mark)**
- (iii) Let $r = 1, S_0 = 1$ and $\sigma^2 = 2$. Suppose that, at time 0, our portfolio consists of a contract with contingent claim $\Phi(S_T) = \log(S_T)$.
Using the pricing formula from (b), find the amount x of stock that we would need to additionally include into our portfolio, in order to make our portfolio delta neutral at time 0. **(4 marks)**

9 Consider the Gai-Kapadia model of debt contagion, on the financial network



with contagion probabilities $\eta_j = \frac{1}{1+j}$.

A brief summary of the Gai-Kapadia model, and associated notation, can be found on the supplementary formula sheet.

- (a) Suppose that bank *A* fails, and defaults on all of its loans. Calculate the probability that the resulting cascade of defaults causes every bank within the network to fail. **(5 marks)**
- (b) Suggest two respects in which the Gai-Kapadia model of debt contagion is not a realistic model. **(2 marks)**

End of Question Paper

MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- $X_n \xrightarrow{d} X \Leftrightarrow$ for any $x \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \mathbb{P}[X_n \leq x] \rightarrow \mathbb{P}[X \leq x]$.
- $X_n \xrightarrow{\mathbb{P}} X \Leftrightarrow$ for any $a > 0$, $\lim_{n \rightarrow \infty} \mathbb{P}[|X_n - X| > a] = 0$.
- $X_n \xrightarrow{a.s.} X \Leftrightarrow \mathbb{P}[X_n \rightarrow X \text{ as } n \rightarrow \infty] = 1$.
- $X_n \xrightarrow{L^p} X \Leftrightarrow \mathbb{E}[|X_n - X|^p] \rightarrow 0$ as $n \rightarrow \infty$.

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants r (discrete interest rate), p_u and p_d (probabilities of stock price increase/decrease), u and d (factors of stock price increase/decrease), and s (initial stock price).

The value of x in cash, held at time t , will become $x(1+r)$ at time $t+1$.

The value of a unit of stock S_t , at time t , satisfies $S_{t+1} = Z_t S_t$, where $\mathbb{P}[Z_t = u] = p_u$ and $\mathbb{P}[Z_t = d] = p_d$, with initial value $S_0 = s$.

When $d < 1+r < u$, the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \quad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time $t = 0, 1, 2, \dots, T$. The case $T = 1$ is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale M_n and a stopping time T , holds if any one of the following conditions is fulfilled:

- (a) T is bounded.
- (b) M_n is bounded and $\mathbb{P}[T < \infty] = 1$.
- (c) $\mathbb{E}[T] < \infty$ and there exists $c \in \mathbb{R}$ such that $|M_n - M_{n-1}| \leq c$ for all n .

MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t dt + G_t dB_t$, and a suitably differentiable function $f(t, x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t) dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u dt + \sigma X_u dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Feynman-Kac formula

Suppose that $F(t, x)$, for $t \in [0, T]$ and $x \in \mathbb{R}$, satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \alpha(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \beta(t, x)^2 \frac{\partial^2 F}{\partial x^2}(t, x) &= 0 \\ F(T, x) &= \Phi(x). \end{aligned}$$

Then $F(t, x) = \mathbb{E}_{t,x}[\Phi(X_T)]$, where X_u satisfies $dX_u = \alpha(u, X_u) dt + \beta(u, X_u) dB_u$.

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t dt$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t dt + \sigma S_t dB_t$, with initial value S_0 .

At time $t \in [0, T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\begin{aligned} \frac{\partial F}{\partial t}(t, s) + rs \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2}(t, s) - rF(t, s) &= 0, \\ F(T, s) &= \Phi(s). \end{aligned}$$

The unique solution F satisfies

$$F(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\Phi(S_T) | \mathcal{F}_t]$$

for all $t \in [0, T]$. Here, the ‘risk-neutral world’ \mathbb{Q} is the probability measure under which S_t satisfies

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G . An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y .

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0, 1]$, we define a model of debt contagion by assuming that:

- (†) For any bank X , with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability η_j the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.