The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **June 2018**

MAS5051 Probability and Probability Distributions **2 hours**

RESTRICTED OPEN BOOK EXAMINATION.

Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.

Candidates should attempt ***All*** *questions.*

The maximum marks for the various parts of the questions are indicated.

The paper will be marked out of 80.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Let X be a continuous random variable with probability density function given by

$$f_X(x) = \begin{cases} k(1-x)^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the value of k. ***(4 marks)***

(ii) Find the distribution function of X. ***(5 marks)***

(iii) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. ***(6 marks)***

(iv) Let $Y = \sqrt{1-X}$, where $\sqrt{1-X}$ denotes the positive square root of $1-X$. Find the probability density function of Y. ***(5 marks)***

2 (i) Suppose one card is randomly drawn from a well-shuffled standard pack of 52 playing cards. Let A be the event that an ace is drawn, B be the event that a black card is drawn and C be the event that a club is drawn.

(a) Give the probabilities of each of the events A, B and C. ***(3 marks)***

(b) Which of the following statements are true? Give reasons for your answers.

(α) A and B are independent.

(β) A and C are independent.

(γ) B and C are independent. ***(6 marks)***

(ii) Suppose that in a certain U.S. state 55% of registered voters are Republicans, 35% are Democrats and 10% are Independents. When these voters were surveyed about increased military spending, 25% of Republicans opposed it, 70% of Democrats opposed it and 55% of Independents opposed it.

(a) What is the probability a randomly selected voter in this state opposes increased military spending? ***(3 marks)***

(b) A registered voter from the state writes a letter to the local paper arguing in favour of increased military spending. What is the probability that this voter is a Democrat? ***(3 marks)***

3 Two random variables U and V have a joint probability mass function as tabulated below.

$p_{U,V}(u,v)$	$u=0$	$u=1$	$u=2$
$v=0$	0.1	0.05	0.3
$v=1$	0.05	0.25	0.25

(i) Find the marginal probability mass functions of U and V. ***(3 marks)***

(ii) If it is known that $U=2$, what is the probability that $V=1$? ***(2 marks)***

(iii) Calculate the means of U and V. ***(3 marks)***

(iv) Calculate the covariance between U and V. ***(4 marks)***

(v) Are U and V independent? Justify your answer. ***(2 marks)***

4 100 patients with gastroesophageal reflux disease are treated with a new drug to relieve pain from heartburn. Following treatment, the time T_i until patient i next experiences pain from heartburn is recorded. It is assumed that the times are independent and identically distributed with probability density function given by

$$f_{T_i}(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) The doctor treating the patients thinks that there is a 50% chance that a random patient will stay pain free for at least 21 days. Show that the value of λ based on the doctor's judgement is 0.0330 (to four decimal places). ***(5 marks)***

(ii) Using the facts that $E(T_i) = \dfrac{1}{\lambda}$ and $\text{Var}(T_i) = \dfrac{1}{\lambda^2}$, find the expectation and variance of

$$\bar{T} = \frac{1}{100}\sum_{i=1}^{100} T_i.$$

(4 marks)

(iii) Explain carefully how to use a Normal approximation to calculate $P(25 \leq \bar{T} \leq 35)$ approximately. You are given the following R output to complete the calculation.

```
> pnorm(1.55,0,1)
[1] 0.9394292
> pnorm(1.75,0,1)
[1] 0.9599408
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(5 marks)

5 (i) Let $\mathbf{Z} = \begin{pmatrix} X \\ Y \end{pmatrix}$ be a random vector with a bivariate normal distribution, with mean vector $\boldsymbol{\mu} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and covariance matrix $\Sigma = \begin{pmatrix} 16 & -5 \\ -5 & 4 \end{pmatrix}$. Let $U = X + Y$ and $V = 2X - Y - 3$. Find the mean vector and covariance matrix of the random vector $\mathbf{W} = \begin{pmatrix} U \\ V \end{pmatrix}$ and hence state the distribution of $\mathbf{W}$.

(9 marks)

(ii) A Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\{1, 2, 3\}$ has transition probability matrix P where

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

and initial distribution $\pi_1^{(0)} = \frac{1}{2}$, $\pi_2^{(0)} = 0$ and $\pi_3^{(0)} = \frac{1}{2}$.

(a) Calculate P^2. Hence, what is $P(X_5 = 1 | X_3 = 3)$? ***(4 marks)***

(b) Calculate the distribution of X_2. ***(4 marks)***

End of Question Paper