The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Financial Mathematics **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) A company borrows £100,000 paying a fixed annual interest rate of 5% compounded continuously. The company will repay the loan in equal monthly payments.

(a) Find the amount of these payments which will repay the loan in 20 years. ***(6 marks)***

(b) Assume now that the company is given the option of repaying the loan with fixed monthly payments paid *forever*. What is the amount of these payments in this case? ***(4 marks)***

(ii) Consider two risk-free, zero-coupon bonds with face value £100. The first matures in 1 year and trades for £99 and the second matures in 2 years and trades for £96.08.

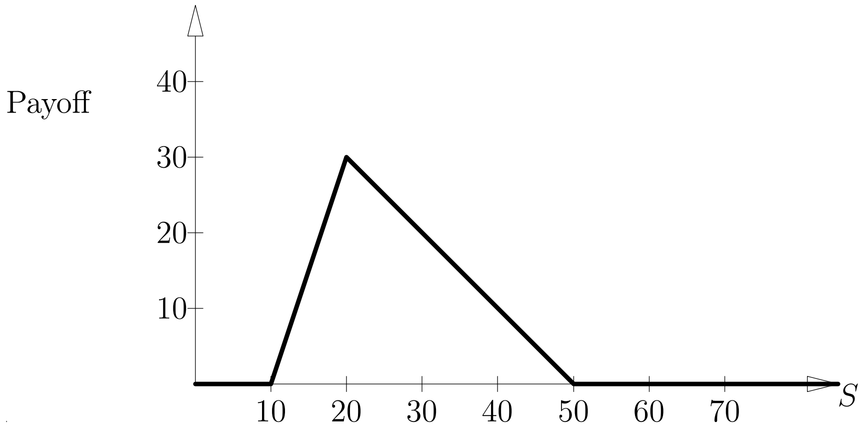
(a) Find the 1-year and 2-year spot interest rates. ***(2 marks)***

(b) Find the forward interest rate for the period starting in 1 year and ending in 2 years. ***(1 mark)***

(c) You are offered an opportunity to borrow or deposit £100,000 for a period of one year starting 1 year hence at an interest rate of 2.5%. Describe in detail an arbitrage opportunity available to you. ***(12 marks)***

2 (i) (a) Describe a portfolio consisting entirely of European call options on the same stock, with same expiration time $T > 0$, but with different strike prices, and whose payoff at time T as a function of S, the spot price of the stock at time T, is described by the graph below.

(6 marks)



(b) Let c_{10}, c_{20} and c_{50} be the prices of the call options above with strike prices $10, 20$ and 50, respectively, and let p_{50} be the price of a European put option on the same stock, with expiration at time T and with strike price 50. By comparing the payoff of the portfolio in (a) and the payoff of the put option above, describe an inequality involving c_{10}, c_{20}, c_{50} and p_{50}. ***(6 marks)***

(ii) Consider two call options, one American and the other European, on the same underlying asset, with same expiration date T and same strike price X. Let S_t denote the underlying asset price at time $0 \leq t \leq T$. Let C_t and c_t denote the prices at time $0 \leq t \leq T$ of the American and European options, respectively. Assume all interest rates are constant and equal to r.

(a) By considering the payoffs at time T of two portfolios consisting of the European call option, the underlying asset and cash, show that for all $0 \leq t \leq T$, $c_t \geq S_t - Xe^{-r(T-t)}$. ***(5 marks)***

(b) Suppose you observe that $C_0 > c_0$. Exhibit an arbitrage opportunity available to you. (You may find it useful to use (a) to deduce that for any $0 \leq t < T$, $c_t > S_t - X$.) ***(8 marks)***

3 (i) (a) State the mathematical definition of Brownian motion. ***(5 marks)***

(b) State Ito's Lemma. ***(3 marks)***

(ii) Assume that a stock price S is given as the Ito process

$$dS = \mu S\,dt + \sigma S\,dB$$

where μ and σ are constants. Assume also that all interest rates are non-stochastic and equal to r. Let Φ be the cumulative normal distribution function. Let $X > 0$ and define $d(S,t) = \dfrac{\log(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$.
Assume henceforth that $f(S,t) = S\Phi(d(S,t))$ satisfies the Black-Scholes PDE.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(a) Compute $\lim_{t\to T,\ t<T} d(S,t)$ and use this to compute $\lim_{t\to T,\ t<T} \Phi(d(S,t))$. (Hint: your answer should depend on the values of S and X.) ***(6 marks)***

(b) Assume that $f(S,t)$ is continuous for all $0 \leq t \leq T$ and $S \geq 0$. Compute $f(0,t)$ for any $0 \leq t < T$, and compute $f(S_T, T)$ for any $S_T \geq 0$. (Hint: use the assumption to write these values as limits.) ***(5 marks)***

(c) An *asset-or-nothing* option with strike price X and expiration time T pays owner at time T: S_T if $S_T > X$, $S_T/2$ if $S_T = X$, and nothing otherwise. Show that the price of this option at time $0 \leq t \leq T$ is $f(S_t, t)$ where S_t is the stock price at time t. *Explain in detail your reasoning.* ***(6 marks)***

4 (i) Consider the following risky investments

Name	expected returns	standard deviation of returns
A Inc.	9%	21%
B Plc.	5%	7%
C Ltd.	15%	36%
D S.A.	12%	15%

(a) Which investments cannot possibly be efficient? ***(4 marks)***

(b) Suppose there is a risk-free return R and you are told that *C Ltd.* is efficient. Find a lower bound for the value of R. ***(8 marks)***

(ii) Consider a world where there are only two risky investments: *Krazy Plc* and *Stolid Inc.* stocks.

	Number of shares	Price per share	Expected return	Standard deviation of return
Krazy Plc	1,000	£20	30%	60%
Stolid Inc.	1,000	£10	8%	10%

The correlation between the returns of these two stocks is 0.5.

(a) What is the market portfolio in this world? ***(2 marks)***

(b) What are the expected return and standard deviation of returns of the market portfolio? ***(3 marks)***

(c) Find the beta coefficient of *Krazy Plc.* ***(4 marks)***

(d) What should be the risk-free return in this world, if one existed? ***(4 marks)***

End of Question Paper