The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017–2018

MAS6340: Analysis I — **2 hours 30 minutes**

Answer **four** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$

1 (i) Let $C_b(\mathbb{R})$ be the complex vector space of all bounded continuous functions $f\colon \mathbb{R} \to \mathbb{C}$. Prove that we have a norm on $C_b(\mathbb{R})$ defined by the formula

$$\|f\| = \sup\{|f(t)| \mid t \in \mathbb{R}\}.$$

(6 marks)

(ii) State the definition of a *Banach space* and a *closed subspace* of a normed vector space. Prove that any closed subspace of a Banach space is complete. ***(6 marks)***

(iii) Prove that the space $C_b(\mathbb{R})$ is a Banach space. ***(9 marks)***

(iv) Let $C_0(\mathbb{R})$ be the subspace of $C_b(\mathbb{R})$ consisting of functions $f \in C_b(\mathbb{R})$ such that $f(t) \to 0$ as $t \to \pm\infty$. Prove that $C_0(\mathbb{R})$ is closed. ***(4 marks)***

2 (i) Let V and W be normed vector spaces. Say what is meant by the statement that a linear map $T\colon V \to W$ is *bounded*, and prove that if T is a bounded linear map, then the kernel, $\ker T$, is closed. ***(5 marks)***

(ii) Let $C^\infty[0,1]$ be the vector space of all real-valued infinitely-differentiable functions $f\colon [0,1] \to \mathbb{R}$, equipped with the norm

$$\|f\| = \sup\{|f(t)| \mid t \in [0,1]\}.$$

Let $C_0^\infty[0,1]$ be the subspace of all infinitely differentiable functions $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = 0$.

Prove that we have a bijective bounded linear map $T\colon C^\infty[0,1] \to C_0^\infty[0,1]$ defined by the formula

$$(Tf)(x) = \int_0^x f(t)\, dt \qquad x \in [0,1].$$

(6 marks)

(iii) Does the map T have a *continuous* inverse? Justify your answer. ***(4 marks)***

(iv) Say what is meant by an open map, and state the open mapping theorem. ***(3 marks)***

(v) Prove that $C_0^\infty[0,1]$ is a closed subspace of the space $C^\infty[0,1]$. Is the normed vector space $C^\infty[0,1]$ complete? Justify your answer. ***(7 marks)***

3 (i) State the Stone-Weierstrass theorem for complex-valued functions. Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let $f_k(z) = z^k$. Show that the span of the set $\{z^k \mid k \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$.

(7 marks)

(ii) Let $C_p[0, 2\pi]$ be the Banach space of continuous functions $f\colon [0, 2\pi] \to \mathbb{C}$ such that $f(0) = f(2\pi)$, under the norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 2\pi]\}.$$

Write down an isometric isomorphism $\alpha\colon C(\mathbb{T}) \to C_p[0, 2\pi]$. ***(3 marks)***

(iii) Let $e_k(t) = \exp(ikt)$. Show that $\{e_k \mid k \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2[0, 2\pi]$. Your answer should include a proof that a dense subset of $C_p[0, 2\pi]$ under the above norm is also a dense subset of $L^2[0, 2\pi]$ under its usual norm. ***(9 marks)***

(iv) Let $f(x) = x$. Find coefficients $a_k \in \mathbb{C}$ such that the series

$$\sum_{k=-\infty}^{\infty} a_k e_k = f$$

in the space $L^2[0, 2\pi]$.

Use the above to evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(6 marks)

4 (i) Let $f, g \in L^1(\mathbb{R})$ be continuous functions. Show that we have a well-defined function $f * g \in L^1(\mathbb{R})$ defined by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

(6 marks)

(ii) For continuous functions $f, g \in L^1(\mathbb{R})$, show that, when taking the Fourier transform, $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$. ***(6 marks)***

(iii) Let $\alpha > 0$. Let $f \in L^1(\mathbb{R})$ be continuous. Let $g(x) = f(\alpha x)$. Show that, when considering the Fourier transform, we have $\hat{g}(\omega) = \dfrac{1}{\alpha}\hat{f}(\dfrac{\omega}{\alpha})$. ***(3 marks)***

(iv) Let $\sigma > 0$. Let

$$g_\sigma = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

Calculate the Fourier transform $\hat{g}_\sigma$. ***(6 marks)***

(v) Show that for any $\sigma, \tau > 0$, we have $g_\sigma * g_\tau = g_{\sqrt{\sigma^2+\tau^2}}$. ***(4 marks)***

You may assume here, if desired, the Fourier inversion theorem.

5 (i) Let V be a Banach space, and let $T: V \to V$ be a bounded linear map. Say what is meant by an *eigenvalue* of T and the *spectrum* of T. Prove that every eigenvalue of T lies in the spectrum. ***(6 marks)***

(ii) Let V be a complex Banach space. Let D be an open subset of $\mathbb{C}$. What is meant by saying a function $f: D \to V$ is *holomorphic*? State *Liouville's theorem* for holomorphic functions with values in V. ***(4 marks)***

(iii) Let $T: V \to V$ be a bounded linear map, and let $D = \mathbb{C} \setminus Spectrum(T)$. Let $R(\lambda) = (T - \lambda I)^{-1}$. Prove that $R(\lambda) - R(\mu) = (\lambda - \mu)R_\lambda R_\mu$ for all $\lambda, \mu \in D$, and that $R: D \to Spectrum(T)$ is holomorphic. You may use the Carl Neumann criterion without proof. ***(8 marks)***

(iv) Use the above to prove that the spectrum of T is always non-empty. Give an example of a bounded linear map with no eigenvalues. ***(7 marks)***

End of Question Paper