



The
University
Of
Sheffield.

MAS6431

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2017-2018

Analytical Dynamics and Classical Field Theory

3 Hours

Answer all questions.

1 (i) Write down the Euler–Lagrange equation for the Lagrangian $L = L(q, \dot{q}, t)$, describing the dynamics of a particle with mass m moving in one dimension with position q and velocity $\dot{q} = dq/dt$ under the influence of a potential $V(q)$. Define all quantities appearing in the equation. **(5 marks)**

(ii) Assume that the Lagrange function L does not explicitly depend on time, i.e. $L = L(q, \dot{q})$. Show that the following equation holds:

$$L - \dot{q} \frac{\partial L}{\partial \dot{q}} = \text{constant.}$$

Hint: Calculate dL/dt .

(8 marks)

(iii) Use the equation above to show that $E = \dot{q}^2/2 + V(q)$ is conserved. What is the interpretation of E ? **(7 marks)**

2 (i) Consider a system with N degrees of freedom, described by a Lagrange–function $L(q_i, \dot{q}_i, t)$ (with $i = 1, \dots, N$). Define the canonical momenta P_i and define the Hamilton–function H . L depends on q_i , \dot{q}_i and time t . What does H depend on?

(6 marks)

(ii) Write down Hamilton’s equations.

(3 marks)

(iii) Show that if the Hamilton–function does not depend explicitly on time, it is constant. What is the interpretation of H in this case?

(3 marks)

(iv) Consider two particles with masses m_1 and m_2 , respectively. They are moving in one dimension under the influence of a potential $V(q_1, q_2) = \kappa_1 q_1^2 + \kappa_2 q_2^2 + \lambda q_1 q_2$, where q_1 is the (generalised) coordinate describing particle 1, q_2 describing particle 2 and κ_1 , κ_2 and λ are constants. Find the Hamilton–function from first principles (i.e. do **not** assume that $H = T + V$, where T is the kinetic energy and V is the potential energy) and write down Hamilton’s equations. Show that the equations of motion for q_1 and q_2 can be written as

$$\ddot{q}_1 + 2\frac{\kappa_1}{m_1}q_1 + \frac{\lambda}{m_1}q_2 = 0 ,$$

$$\ddot{q}_2 + 2\frac{\kappa_2}{m_2}q_2 + \frac{\lambda}{m_2}q_1 = 0 .$$

(8 marks)

3 Consider the Lagrangian of a complex scalar field $\Phi(x)$, given by

$$\mathcal{L} = \eta^{\mu\nu}(\partial_\mu\Phi^*)(\partial_\nu\Phi) - V(\Phi\Phi^*),$$

where V is an arbitrary function depending only on the product $\Phi\Phi^*$.

(i) Show that the Lagrangian above is invariant under the transformation $\Phi \rightarrow \Phi' = e^{i\theta}\Phi$, where θ is constant. (4 marks)

(ii) Show that the equations of motion for Φ^* and Φ are given by

$$\begin{aligned} \partial^\mu\partial_\mu\Phi^* &= -\frac{\partial V}{\partial\Phi}, \\ \partial^\mu\partial_\mu\Phi &= -\frac{\partial V}{\partial\Phi^*}. \end{aligned}$$

Hint: Treat Φ and Φ^* as independent variables. (6 marks)

(iii) State Noether's theorem and argue that, because of part (i), there must be a conserved current. You don't have to write down the equation for the current. (3 marks)

(iv) You are given that the Noether current is

$$J_\mu = \Phi^*(\partial_\mu\Phi) - (\partial_\mu\Phi^*)\Phi.$$

Show that $\partial^\mu J_\mu = 0$, provided that Φ and Φ^* satisfy the equations of motion given above. (7 marks)

4 For this question, you are given that the Christoffel–symbols are given by

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2}g^{\alpha\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right).$$

(i) Consider the following line–element:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2(\sin^2 \theta d\phi^2 + d\theta^2),$$

where the coordinates are $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and $\alpha(r)$ and $\beta(r)$ are undetermined functions of r . Write down all non–vanishing components of the metric tensor $g_{\mu\nu}$. Write down all non–vanishing components of the inverse metric $g^{\mu\nu}$ as well. **(6 marks)**

(ii) Show that

$$\begin{aligned} \Gamma_{tt}^r &= e^{2(\alpha-\beta)} \frac{\partial \alpha(r)}{\partial r} \\ \Gamma_{\phi\phi}^{\theta} &= -\sin \theta \cos \theta \\ \Gamma_{\phi\phi}^r &= -re^{-2\beta} \sin^2 \theta \\ \Gamma_{tr}^t &= \frac{\partial \alpha}{\partial r}. \end{aligned}$$

(7 marks)

(iii) You are given the following two components of the Ricci–tensor:

$$\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)} \left[\frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r} \right)^2 - \frac{\partial \alpha}{\partial r} \frac{\partial \beta}{\partial r} + \frac{2}{r} \frac{\partial \alpha}{\partial r} \right], \\ R_{rr} &= -\frac{\partial^2 \alpha}{\partial r^2} - \left(\frac{\partial \alpha}{\partial r} \right)^2 + \frac{\partial \alpha}{\partial r} \frac{\partial \beta}{\partial r} + \frac{2}{r} \frac{\partial \beta}{\partial r}. \end{aligned}$$

Assume that the energy–momentum tensor of matter vanishes (locally). What does that imply for R_{tt} and R_{rr} ? Furthermore, show that in this case

$$\alpha(r) = -\beta(r) + C,$$

where C is a constant.

(7 marks)

5 For this question, you are given that the covariant derivative of a four-vector V_α is $V_{\alpha;\beta} = \partial_\beta V_\alpha - \Gamma_{\alpha\beta}^\mu V_\mu$. You are also given that under the coordinate transformation $x^\mu \rightarrow x'^\mu$ the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) . \quad (1)$$

(i) Show that if $V_{\alpha;\beta} = 0$ holds in one coordinate system, it holds in all coordinate systems. [Hint: no explicit calculation is needed for this question! What is the transformation rule for the covariant derivative under coordinate transformations?]

(2 marks)

(ii) Consider an infinitesimal coordinate transformation of the form $x'^\mu = x^\mu + \epsilon v^\mu(x)$, with $\epsilon \ll 1$ and v^μ is a vector-field.

(a) Show that, to first order in ϵ

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta_\nu^\mu - \epsilon \frac{\partial v^\mu}{\partial x^\nu} . \quad (2)$$

(4 marks)

(b) Use equations (1) and (2) to show that, to first order in ϵ

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \epsilon \left(g_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} + g_{\mu\beta} \frac{\partial v^\beta}{\partial x^\nu} \right) . \quad (3)$$

(6 marks)

(c) Show that if $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$, the vector field v^α obeys

$$v^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} + g_{\mu\beta} \frac{\partial v^\beta}{\partial x^\nu} = 0 .$$

Hint: You need to use eq. (3) and a Taylor expansion of $g'_{\mu\nu}(x')$ for x' close to x .

(8 marks)

End of Question Paper