The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017–2018

Mathematical Methods And Modelling Of Natural Systems — 2 hours

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

1 A function $f(x)$ is defined for $-\infty < x < \infty$ by

$$f(x) = x\, e^{-x^2}.$$

(a) Show that for real k the Fourier transform, $\hat{f}(k)$, of $f(x)$ is given by

$$\hat{f}(k) = \frac{1}{2}\, i\sqrt{\pi}\, k\, e^{-k^2/4}.$$ *(8 marks)*

[You may assume that

$$\int_{-\infty}^{\infty} (x-a)\, e^{-(x-a)^2}\, dx = \int_{-\infty}^{\infty} x\, e^{-x^2}\, dx$$

and

$$\int_{-\infty}^{\infty} e^{-(x-a)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

where a may be complex, but does not depend on x.]

(b) Show that applying the inverse Fourier transform to $\hat{f}(k)$ gives $f(x)$.
(8 marks)

(c) Verify that Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2\, dk$$

holds for this $f(x)$. *(9 marks)*

2 The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) Find, by explicit integration, the Laplace transform of $\cos t$ for $\operatorname{Re} s > 0$.

(5 marks)

(b) The function $f(t)$ is defined by

$$f(t) = \begin{cases} t & 0 \leqslant t \leqslant 1 \\ 0 & t > 1. \end{cases}$$

Find the Laplace transform of $f(t)$. ***(5 marks)***

(c) $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Find $F(t)$ for $t > 0$ if $f(t)$ is as defined in part (b), and $g(t) = \cos t$.

(11 marks)

(d) Given that for $t > 0$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\},$$

use the results of parts (a), (b) and (c) to deduce that the inverse Laplace transform of

$$\frac{1 - e^{-s} - se^{-s}}{s(1+s^2)}$$

is

$$H(1-t) - \cos t + H(t-1)\{\sin(1-t) + \cos(1-t)\} \quad \text{for } t > 0,$$

where H is the Heaviside step function. ***(4 marks)***

3 The diffusion of a contaminant in a one-dimensional space satisfies the partial differential equation

$$\frac{\partial \Gamma}{\partial t} = \kappa \frac{\partial^2 \Gamma}{\partial x^2} \qquad \text{for } -\infty < x < \infty, \qquad t \geqslant 0,$$

where $\Gamma(x,t)$ is the contaminant concentration at position x and time t, and κ is a positive constant.

The Fourier transform of $\Gamma(x,t)$ with respect to x is defined by

$$\hat{\Gamma}(k,t) = \int_{-\infty}^{\infty} e^{ikx}\,\Gamma(x,t)\,dx.$$

Show that

$$\hat{\Gamma}(k,t) = \hat{\Gamma}(k,0)e^{-\kappa k^2 t}.$$ ***(5 marks)***

(a) The initial condition is

$$\Gamma(x,0) = \begin{cases} 1 - x^2 & |x| \leqslant 1 \\ 0 & |x| > 1. \end{cases}$$

Find $\hat{\Gamma}(k,0)$, and show that

$$\Gamma(x,t) = \frac{1}{\sqrt{1+4\kappa t}} \exp\left(-\frac{x^2}{1+4\kappa t}\right).$$ ***(12 marks)***

[You may assume that

$$\int_{-\infty}^{\infty} e^{-b^2(x-a)^2}\,dx = \frac{1}{b}\sqrt{\pi},$$

where b is real, and a may be complex, but does not depend on x.]

(b) The initial condition is

$$\Gamma(x,0) = \sqrt{\pi}\,\delta(x),$$

where $\delta(x)$ is the Dirac delta function.

Find $\Gamma(x,t)$, and show that as $t \to \infty$ the answer in (a) gives the same result. ***(8 marks)***

4 The function $y(x)$ satisfies the ordinary differential equation

$$x^2 y'' + 3xy' - 3y = x^3 e^{-x}$$

in $0 < x < \infty$, with the boundary conditions that y is finite at $x = 0$ and as $x \to \infty$.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2 y'' + 3xy' - 3y = 0.$$ ***(3 marks)***

(b) Given that Green's function, $G(x;\xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} -\dfrac{x}{4\xi^2} & 0 \leqslant x < \xi \\ \\ -\dfrac{\xi^2}{4x^3} & \xi < x < \infty. \end{cases}$$ ***(10 marks)***

(c) Using Green's function, show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = \frac{30}{x^3} + e^{-x}\left(x + 5 + \frac{15}{x} + \frac{30}{x^2} + \frac{30}{x^3}\right).$$ ***(12 marks)***

[You may assume that for $m \geqslant 1$

$$\int \xi^m e^{-\xi}\, d\xi = -\xi^m e^{-\xi} + m \int \xi^{m-1} e^{-\xi}\, d\xi.$$]

5 Consider the equation

$$(1-\epsilon)x^3 + 4x^2 + x - 6 = 0, \qquad (1)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (1) can be written as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots ,$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to show that one of the three solutions to equation (1) is

$$x = -3 - \frac{27}{4}\epsilon + \cdots ,$$

and to find the other two solutions, correct to order ϵ as $\epsilon \to 0$.

(14 marks)

(b) Given the rearrangement

$$x = -2 + \epsilon \frac{x^3}{(x-1)(x+3)}$$

of (1), use iteration to find the solution close to -2, correct to order ϵ^2 as $\epsilon \to 0$.

(11 marks)

End of Question Paper