The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring 2018-19**

FOUNDATION YEAR CORE MATHEMATICS **3 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which ***must be answered on the exam paper itself****.*

Answers to Section B must be written on the answer booklet provided.

Total marks: 100. ***NO CALCULATORS ALLOWED.***

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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Section A

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer ***on the question paper****. Total marks for this section:* ***(40 marks)***

A1 The statement that $\left(-\frac{1}{6}\right)^p \in \mathbb{R}$ is true if and only if

A. $p \in \mathbb{R}$; **B.** $p \in \mathbb{Q}$; **C.** $p \in \mathbb{Z}$; **D.** $p \in \mathbb{N}$.

A2 Let y, $z > 0$. The expression $yz^{y \log_z\left(\left(\frac{1}{y}\right)^y\right)} =$

A. y^{1-y^2}; **B.** -1; **C.** $\frac{1}{y}$; **D.** $-y^2$.

A3 Let $\beta > 0$. The expression $\frac{\log_{64} \beta}{\log_2 \beta} =$

A. $\log_{62} \beta$; **B.** 32; **C.** $\frac{1}{6}$; **D.** -6.

A4 For $c > 1$, as $t \to \infty$, $\log_c t \to$

A. 0; **B.** 1;
C. a positive constant; **D.** ∞.

A5 The expression $\sum_{m=1}^{M} m$ represents

A. an arithmetic series; **B.** an infinite series;
C. a geometric series; **D.** a binomial series.

A6 The result, S_M of the series given in question **A5** above is

A. $\frac{M(M-1)}{2}$; **B.** $M(M-1)$; **C.** $\frac{M(M+1)}{2}$; **D.** $M!$.

A7 An alternative way to write the result S_M of the series given in question **A5** above is

A. $(M+1)!M$; **B.** $M(M-1)!$; **C.** $\binom{M-1}{M+1}$; **D.** $\binom{M+1}{2}$.

A8 The series given in question **A5** above can be expressed as

A. $\sum_{m=0}^{M} \frac{(m+1)!}{m!}$; **B.** $\sum_{k=0}^{M+1} \frac{(k+1)!}{k!}$;

C. $\sum_{l=0}^{M-1} \frac{(l+1)!}{l!}$; **D.** $\sum_{m=0}^{M-1} \frac{m!}{(m-1)!}$

A9 Which one of the results below corresponds to $\frac{1}{2}\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$?

A. $\frac{\sqrt{3}}{4}$; **B.** $\frac{\sqrt{2}}{8}$; **C.** $\sqrt{2}$; **D.** $\frac{2}{\sqrt{2}}$.

A10 Which one of the results below corresponds to $\cot\left(\frac{5\pi}{6}\right)$?

A. $-\sqrt{3}$; **B.** $-\frac{\sqrt{3}}{3}$; **C.** $\frac{\sqrt{3}}{3}$; **D.** $\sqrt{3}$.

A11 Given that $\sin^2\lambda + \cos^2\lambda = 1$ for all $\lambda \in \mathbb{R}$, which one of the following statements is true for all $0 < \lambda < \frac{\pi}{2}$?

A. $\operatorname{cosec} 2\lambda = \frac{\cot\lambda}{2}$; **B.** $\operatorname{cosec}\lambda = \sin\lambda - \cot\lambda\cos\lambda$;

C. $\operatorname{cosec} 2\lambda = \frac{1}{2}\left(\tan\lambda + (\tan\lambda)^{-1}\right)$; **D.** $\operatorname{cosec}\lambda = \cot\lambda\cos\lambda + \sin\lambda$.

A12 Which one of the following statements is true for all $\delta \neq k \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$?

A. $\tan^2\delta - \sec^2\delta = 1$; **B.** $\sec^2\delta + \operatorname{cosec}^2\delta = 1$;

C. $1 - \operatorname{cosec}^2\delta = \cot^2\delta$; **D.** $\operatorname{cosec}^2\delta - \cot^2\delta = 1$.

A13 An unbiased dodecahedral die has faces labelled by the first twelve prime numbers. Let A correspond to rolling the die and obtaining a number $n \geqslant 27$. Given this information, which one of the following statements is true?

A. $\mathrm{P}(A) = 4$; **B.** $\mathrm{P}(\overline{A}) = 9$; **C.** $\mathrm{P}(A) = \dfrac{3}{4}$; **D.** $\mathrm{P}(\overline{A}) = 0.75$.

A14 Suppose that $\mathrm{P}(E_1) = 0.2$, $\mathrm{P}(E_2) = 0.4$ and $\mathrm{P}(E_1 \cap E_2) = 0.1$. What is $\mathrm{P}(E_2|E_1)$?

A. 4; **B.** 2; **C.** $\dfrac{1}{4}$; **D.** $\dfrac{1}{2}$.

A15 Two standard unbiased tetrahedral dice are rolled simultaneously. What is the probability of obtaining different numbers on either die?

A. $\dfrac{3}{4}$; **B.** $\dfrac{1}{4}$; **C.** $\dfrac{3}{16}$; **D.** $\dfrac{1}{16}$.

A16 Let X and Y be two mutually exclusive events. Which one of the following statements is always true?

A. $\mathrm{P}(X \cap Y) = \mathrm{P}(X) + \mathrm{P}(Y)$; **B.** $\mathrm{P}(X \cup Y) = \mathrm{P}(X) \cdot \mathrm{P}(Y)$;

C. $\mathrm{P}(X) = \mathrm{P}(X \cup Y) - \mathrm{P}(Y)$; **D.** $\mathrm{P}(\overline{X}) = \mathrm{P}(Y)$.

A17 Which one of the following expressions is meaningless?

A. $\dfrac{0}{0}$; **B.** $|\boldsymbol{a}|\boldsymbol{b}$; **C.** $\boldsymbol{a}\boldsymbol{b}$; **D.** $\dfrac{x}{0}$, $(x \neq 0)$.

A18 Which one of the following expressions has a result that is undefined over the reals?

A. $\sqrt[7]{0}$; **B.** $\sqrt[6]{-1}$; **C.** $\sqrt[5]{-1}$; **D.** $\sqrt[4]{0}$.

A19 If $\boldsymbol{x} = -\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{y} = -\boldsymbol{i} + \dfrac{1}{100}\boldsymbol{j}$ and $k \in \mathbb{R}$, then $\displaystyle\sum_{n=1}^{\infty} k\,(\boldsymbol{x} \bullet \boldsymbol{y})^n$

A. is meaningless; **B.** converges;

C. diverges; **D.** is a finite series.

A20 If $f(t) = \sum_{j=1}^{5} \binom{5}{j} (\boldsymbol{b} \bullet \boldsymbol{v})^{5-j}\, t^j$ then $f(t)$

A. is meaningless;

B. is a binomial expansion of $(\boldsymbol{b} \bullet \boldsymbol{v} + t)^5$;

C. is a finite geometric series;

D. $= (\boldsymbol{b} \bullet \boldsymbol{v} + t)^5 - (\boldsymbol{b} \bullet \boldsymbol{v})^5$.

A21 The derivative of $y = x^3 - 4x^{-1} + x^{\frac{3}{7}} + \sqrt[6]{x} - \ln(x) + e^{\pi}$ is $y' =$

A. $3x^2 + 4x^{-2} + \frac{3}{7}x^{-\frac{4}{7}} + \frac{1}{6}x^{-\frac{5}{6}} - \frac{1}{x} + e^{\pi x}$;

B. $3x^2 + 4x + \frac{7}{3}x^{-\frac{4}{7}} + \frac{1}{6}x^{-\frac{5}{6}} - \frac{1}{x} + e^{\pi}$;

C. $3x^2 - 4x^{-2} + \frac{3}{7}x^{-\frac{4}{7}} - 6x^{-7} - \frac{1}{x}$;

D. $3x^2 + 4x^{-2} + \frac{3}{7}x^{-\frac{4}{7}} + \frac{1}{6}x^{-\frac{5}{6}} - \frac{1}{x}$.

A22 The derivative of $y = 5\sin(\tan^{-1}(x))$ is $y' =$

A. $5\cos(\tan^{-1}(x))$;

B. $-5\cos(\tan^{-1}(x))\operatorname{cosec}^2(x)$;

C. $\dfrac{5\cos(\tan^{-1}(x))}{1+x^2}$;

D. $\dfrac{5\sin(x)}{x^2+1}$.

A23 The derivative of $y = (x^7 + 1)(\cot(x))$ is $y' =$

A. $7x^6\cot(x) + (x^7+1)\sec^2(x)$;

B. $7x^6\cot(x) - (x^7+1)\operatorname{cosec}^2(x)$;

C. $7x^6\cot(x) - (x^7+1)\sec^2(x)$;

D. $7x^6\cot(x) + (x^7+1)\operatorname{cosec}^2(x)$.

A24 The derivative of $y = \dfrac{\sin^{-1}(x)}{e^x}$ is $y' =$

A. $\dfrac{\sqrt{1-x^2} - (1-x^2)\sin^{-1}(x)}{(1-x^2)e^x}$;

B. $\dfrac{\cos^{-1}(x) - \sin^{-1}(x)}{e^x}$;

C. $\dfrac{\sqrt{x^2-1} - (x^2-1)\sin^{-1}(x)}{(x^2-1)e^x}$;

D. $\dfrac{\sqrt{1-x^2} + (1-x^2)\sin^{-1}(x)}{(1-x^2)e^x}$.

A25 $\int x^2 - 7x^{-\frac{2}{3}} + \sqrt[9]{x} + \frac{1}{x} + \pi + 2e\,dx =$

A. $\frac{1}{3}x^3 - 21x^{\frac{1}{3}} + \frac{9}{10}x^{\frac{10}{9}} + \ln(x) + (\pi + 2e)x + c;$

B. $\frac{1}{3}x^3 - 21x^{\frac{1}{3}} + \frac{9}{10}x^{\frac{10}{9}} + \ln|x| + c;$

C. $\frac{1}{3}x^3 - 21x^{\frac{1}{3}} + \frac{9}{10}x^{\frac{10}{9}} + \ln|x| + (\pi + 2e)x + c;$

D. $\frac{1}{3}x^3 - 21x^{\frac{1}{3}} + \frac{9}{10}x^{\frac{10}{9}} + \ln|x| + (\pi + 2e)x.$

A26 The equations of the tangent, $t(x)$, and the normal, $n(x)$, of $y = \ln(x+e) + e^{-x}$ at $x = 0$ are

A. $t(x) = \left(\frac{1+e}{e}\right)x + 1,$ and $n(x) = -\frac{e}{e+1}x + 1;$

B. $t(x) = \left(\frac{1}{x+e} - e^{-x}\right)x + 2,$ and $n(x) = -\left(-x - e + e^x\right)x + 2;$

C. $t(x) = \left(\frac{1+e}{e}\right)x + 2,$ and $n(x) = -\frac{e}{e+1}x + 2;$

D. $t(x) = \left(\frac{1-e}{e}\right)x + 2,$ and $n(x) = \frac{e}{e-1}x + 2.$

A27 $\int e^x \sin(2x)\,dx =$

A. $e^x(\sin(2x) - \cos(2x)) + c;$

B. $\frac{1}{2}e^x(\sin(2x) - \cos(2x)) + c;$

C. $e^{2x}(\sin(2x) + \cos(2x)) + c;$

D. $\frac{1}{5}e^x(\sin(2x) - 2\cos(2x)) + c.$

A28 $\int 2x\cos(x^2)e^{\sin(x^2)}\,dx =$

A. $e^{\sin(x^2)} + c;$

B. $x^2\sin(x^2)e^{\sin(x^2)} + c;$

C. $e^{\cos(x^2)} + c;$

D. $2xe^{\sin(x^2)} - 2e^{\sin(x^2)} + c.$

A29 $\int_{-1}^{2} x^2 - x^3\,dx =$

A. $-\frac{5}{4}$;   **B.** $-\frac{4}{3}$;   **C.** $\frac{5}{4}$;   **D.** $-\frac{3}{4}$.

A30 $\int_{-1}^{1} xe^{-x}\,dx =$

A. $\frac{2}{e}$;   **B.** e;   **C.** $-\frac{2}{e}$;   **D.** 1.

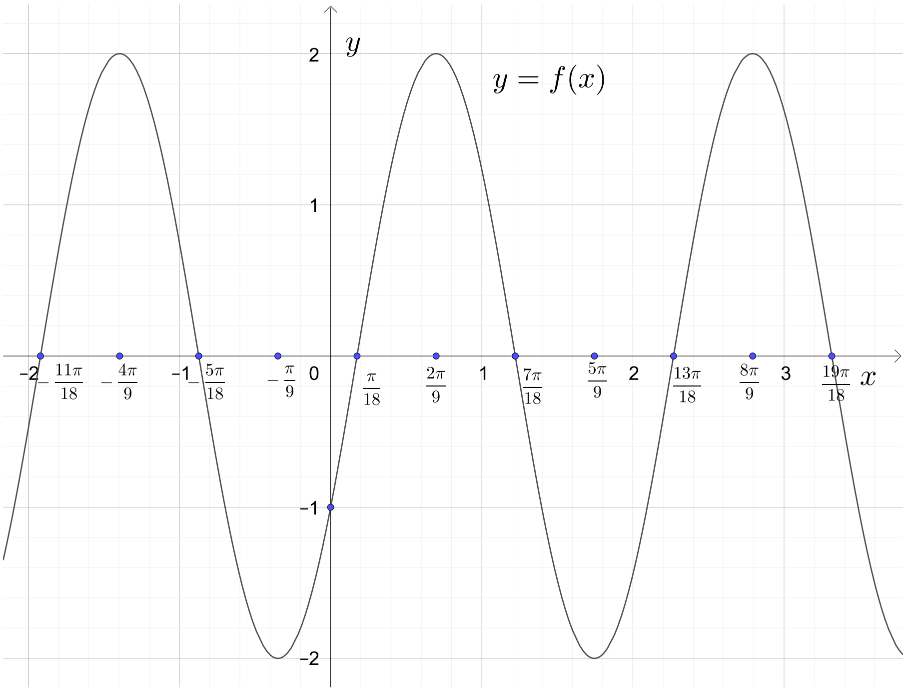
A31 Let $f(x)$ be an odd function and a be a real number. Assume that $\int_{-a}^{a} f(x)\,dx$ exists. Then $\int_{-a}^{a} f(x)\,dx =$

A. 0;   **B.** $\int_{0}^{2a} f(x)\,dx$;   **C.** $2\int_{0}^{a} f(x)\,dx$;   **D.** 1.

A32 Let ϵ be any very small positive real number. A stationary point $P = (a, f(a))$ of the curve $y = f(x)$ is a point of inflection if and only if

A. $f''(a) = 0$;
B. $f''(a) = 0$ and $f'(a-\epsilon) = f'(a+\epsilon)$ are both either positive or negative;
C. $f'(a-\epsilon)$ and $f'(a+\epsilon)$ are both either positive or negative;
D. $f''(a) = f'''(a) = 0$.

A33 Which function $y = f(x)$ has the following graph?



A. $y = f(x) = 2\sin\left(3x - \dfrac{\pi}{18}\right)$; **B.** $y = f(x) = 2\sin\left(3x - \dfrac{\pi}{6}\right)$;

C. $y = f(x) = 2\sin\left(2x - \dfrac{\pi}{6}\right)$; **D.** $y = f(x) = 3\sin\left(3x - \dfrac{\pi}{6}\right)$.

A34 If $x = \dfrac{t-1}{e^t}$ and $y = \dfrac{t^2+1}{e^t}$ then $\dfrac{dy}{dx} =$

A. $\dfrac{t^2-2t-1}{t-2}$ for $t \neq 2$; **B.** $\dfrac{t-2}{(t-1)^2}$ for $t \neq 1$;

C. $\dfrac{(t-1)^2}{t-2}$ for $t \neq 2$; **D.** $\dfrac{t-2}{t^2-2t-1}$ for $t^2 - 2t - 1 \neq 0$.

A35 $\displaystyle\int \sin^7(x+1)\,dx =$

A. $\dfrac{1}{7}\cos^7(x+1) + c$;

B. $-\cos(x+1) + \cos^3(x+1) - \dfrac{3}{5}\cos^5(x+1) + \dfrac{1}{7}\cos^7(x+1) + c$;

C. $\cos(x+1) + \cos^3(x+1) - \dfrac{3}{5}\cos^5(x+1) + \dfrac{1}{7}\cos^7(x+1) + c$;

D. $\cos(x+1) + \cos^3(x+1) + \dfrac{3}{5}\cos^5(x+1) + \dfrac{1}{7}\cos^7(x+1) + c$.

A36 The area between under the curve $y = x^2 - 4x + 3$ and the lines $x = \dfrac{5}{2}$ and $x = 4$ is

A. $\dfrac{9}{8}$; **B.** e; **C.** π; **D.** $\dfrac{37}{24}$.

A37 If $y = x^{\cos(3x-1)}$ then $\dfrac{dy}{dx}$ is

A. $-3\sin(3x-1)x^{\cos(3x-1)}$;
B. $\dfrac{-3\sin(3x-1)}{x} \times x^{\sin(3x-1)}$;
C. $\dfrac{(\cos(3x-1) - 3x\sin(3x-1)\ln x)x^{\cos(3x-1)}}{x}$;
D. $\dfrac{3x}{3x-1} \times x^{\cos(3x-1)}$.

A38 $\displaystyle\int_1^2 x(3x^2-1)^{\frac{1}{2}}\,dx =$

A. $\dfrac{1}{9}(11\sqrt{11} - 2\sqrt{2})$; **B.** $\dfrac{2}{3}(11\sqrt{11} - 2\sqrt{2})$;
C. $\dfrac{1}{9}(\sqrt{11} - \sqrt{2})$; **D.** $-\dfrac{2}{9}(11\sqrt{11} - 2\sqrt{2})$.

A39 If $f(x) = \dfrac{x}{x^2-2} + 2$ then

A. $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to-\infty} f(x) = 0^-$;
B. $\lim_{x\to\infty} f(x) = 2^-$ and $\lim_{x\to-\infty} f(x) = 2^+$;
C. $\lim_{x\to\infty} f(x) = 2^+$ and $\lim_{x\to-\infty} f(x) = 2^-$;
D. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

A40 $\displaystyle\int \frac{4x^3 - x + 1}{x(x+1)(x^2+1)}\,dx =$

A. $2\ln|x| + \ln|x-1| + \ln(x^2+1) + \tan^{-1}(x) + c$;
B. $\ln|x| + \ln|x+1| + \ln(x^2+1) - 3\tan^{-1}(x) + c$;
C. $\tan^{-1}(x^2+1) + \tan^{-1}(x^2+x)$;
D. $\ln(x) + \ln(x+1) + \ln(x^2+1) - 3\tan^{-1}(x) + c$.

Section B

Give your full solution in the problem book for each of the following questions. Total marks for this section: ***(60 marks)***

B1 (i) Let b be the brightness of a light source perceived by the human eye. Let I, measured in units $\mathrm{W \cdot m^{-2}}$, be the corresponding intensity of light emitted by the light source.

Fechner's law states that

$$b = C \ln\left(\frac{I}{I_0}\right),$$

where C and I_0 are constants.

Assuming that $I_0 = 1\ \mathrm{W \cdot m^{-2}}$, if light source α is perceived by the human eye to be twice as bright as light source β, then using Fechner's law, determine how the intensity of light emissions from α are related to those from β.

(3 marks)

(ii) Solve $\log_y(2-x) - \log_y(3+x) = 2$ for y, where $-3 < x < 2$. ***(3 marks)***

B2 (i) (a) Write down a general formula for

$$\sum_{b=1}^{B} \left(-\frac{2}{3}\right)^{b-2},$$

simplified as far as possible.

(b) Use your formula from part (a) to show that the sum of the first five terms of the series is $-\dfrac{55}{54}$.

(c) Find $\displaystyle\sum_{b=1}^{B} \left(-\frac{2}{3}\right)^{b}$ as $B \to \infty$. ***(5 marks)***

(ii) Find the coefficients of the second and third terms obtained when the expression

$$\left(\frac{y}{2} + 3x\right)^7$$

is expanded in descending powers of y, showing *all* your working.

Note: You may not use Pascal's triangle to reach your solution.

(3 marks)

B3 The probability distribution of a discrete random variable, K, is as follows:

k	1	2	3
P_k	0.25	0.35	0.40

where P_k denotes the probability that $K = k$.

Find $\sigma(K)$, simplifying your answer as far as possible.

Note: You may use the fact that 251 is prime. ***(4 marks)***

B4 Let $j(\alpha) = \cos\alpha + \sqrt{3}\sin\alpha$.

(i) Write $j(\alpha)$ in the form

$$j(\alpha) = N\cos(\alpha + s), \qquad N \text{ is a constant and } 0 < s < \frac{\pi}{2}.$$

(4 marks)

(ii) Determine the minimum value of

$$J(\alpha) = 6 - 6j(\alpha)$$

and the values of α for which this minimum occurs. ***(2 marks)***

B5 Let $\boldsymbol{m} = \sqrt{\dfrac{a+\sqrt{b}}{2}}\,\boldsymbol{i} + \sqrt{\dfrac{a-\sqrt{b}}{2}}\,\boldsymbol{j}$, and let $\boldsymbol{\mu} = \sqrt{\dfrac{a+\sqrt{b}}{2}}\,\boldsymbol{i} - \sqrt{\dfrac{a-\sqrt{b}}{2}}\,\boldsymbol{j}$, where $a^2 > b > 0$.

(i) Express $\boldsymbol{m} \bullet \boldsymbol{\mu}$ in terms of a and/or b. ***(2 marks)***

(ii) Let ϕ be the angle between $\boldsymbol{m}$ and $\boldsymbol{\mu}$.

(a) Give a formula for ϕ.

(b) Now let $a^2 = 2b$. Using your formula for ϕ in part (a), find ϕ.

(4 marks)

B6 Let $y = f(x) = \frac{1}{3}x^3 - \ln(x^2+1)$.

(i) Find the stationary points of $f(x)$ and determine their nature. ***(9 marks)***

(ii) Sketch the graph of $y = f(x)$, stating the domain of the function and clearly marking or stating the x- and y-intercepts, stationary points, behaviour where the function is not defined and the behaviour of $f(x)$ as $x \to \pm\infty$, if applicable.
You are given the information that the function has (exactly) two x-intercepts, one of them at approximately 1.54. ***(6 marks)***

(iii) Find the *area* enclosed by $y = \frac{1}{3}x^3 - \ln(x^2+1)$, the x-axis, the y-axis and $x = \frac{\sqrt{3}}{3}$, simplifying your answer as much as possible. ***(7 marks)***

B7 Differentiate $y = 3x^2 - x + x^{-1}$ from *First Principles*, showing clearly the definition and your workings. ***(6 marks)***

B8 Find $\int \frac{x-1}{3x^2-6x-14}\,dx$. ***(2 marks)***

End of Question Paper