The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Mathematics Core 1 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: ***55***

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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Section A:

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(26 marks)***

A1 Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Which of the following options is an element of $A \times B$?

A. $(2, 1)$ **B.** $\{2, 2\}$ **C.** $\{(2, 2)\}$ **D.** $(2, 2)$

A2 The number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is

A. 2^{10} **B.** 10^2 **C.** 2^5 **D.** $(2 \times 10)^{10}$

A3 How many different functions from $\{x, y, z\}$ to $\{a, b\}$ are there?

A. 6 **B.** 8 **C.** 9 **D.** $\binom{3}{2}$

A4 How many different anagrams does `ABCDEABCDE` have?

A. $\frac{10!}{2^5}$ **B.** $\frac{10!}{5!}$ **C.** $\frac{10!}{5^2}$ **D.** $\frac{10^{10}}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

A5 If $r \neq 1$ then $r^2 + r^4 + r^6 + r^8 + r^{10}$ is equal to

A. $\frac{r^2(1 + r^{10})}{1 + r^2}$ **B.** $\frac{r^2(1 - r^{10})}{1 - r^2}$ **C.** $\frac{r^2(1 + r^2)^5}{1 + r^5}$ **D.** $\frac{1 - r^{10}}{1 - r^2}$

A6 If $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{3}{5}$ then $\sin(2\theta) =$

A. $\frac{6}{25}$ **B.** $\frac{12}{25}$ **C.** $\frac{18}{25}$ **D.** $\frac{24}{25}$

A7 If $\tan x = \dfrac{1}{2}$ and $\tan y = \dfrac{1}{3}$ then $\tan(x+y) =$

A. 0 **B.** 1 **C.** 2 **D.** 3

A8 $\displaystyle\lim_{x \to 0} \frac{e^{2x} - 1}{x} =$

A. 2 **B.** 1 **C.** 0 **D.** ∞

A9 The function $f : \mathbb{R} \to \mathbb{R}$ has the property that $x^2 \leq f(x) \leq x^2 + 1$ for $x > 0$. Then $\displaystyle\lim_{x \to \infty} \frac{f(x)}{x^2} =$

A. 0 **B.** 1 **C.** 2 **D.** ∞

A10 If $e^y = \sin x$ for $0 < x < \pi$ then $\dfrac{dy}{dx} =$

A. $\tan x$ **B.** $\cot x$ **C.** $-\tan x$ **D.** $3 - \cot x$

A11 The tangent to the curve $x^3 + y^3 = 2$ at the point $(1, 1)$ has slope

A. 1 **B.** 0 **C.** -1 **D.** ∞

A12 $\displaystyle\int_0^2 \frac{x^2}{1 + x^3}\, dx =$

A. 0 **B.** $\dfrac{2}{3}$ **C.** $\ln(3)$ **D.** $\dfrac{2}{3}\ln(3)$

A13 $\displaystyle 1 + \int_1^2 \ln(x)\, dx =$

A. 2 **B.** $\ln(2)$ **C.** $\ln(3)$ **D.** $2\ln(2)$

A14 $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx =$

A. π **B.** $\frac{\pi}{2}$ **C.** $\frac{\pi}{3}$ **D.** $\frac{\pi}{4}$

A15 $\displaystyle\frac{1-3i}{1+2i} =$

A. $-1-i$ **B.** $5-i$ **C.** $1-i$ **D.** $1+i$

A16 If $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{5}$ then $e^{i\theta} =$

A. $\frac{4+3i}{5}$ **B.** $\frac{4-3i}{5}$ **C.** $\frac{-4+3i}{5}$ **D.** $\frac{-4-3i}{5}$

A17 If $z = 2+i$ then $|z^4| + |\overline{z}^4| =$

A. 25 **B.** 50 **C.** 1000 **D.** 1250

A18 If $\arg(z) = \frac{\pi}{3}$ and $\arg(w) = \frac{\pi}{4}$, then $\arg\left(\frac{z}{w}\right) =$

A. $\frac{7\pi}{12}$ **B.** $\frac{\pi}{5}$ **C.** $\frac{4}{3}$ **D.** $\frac{\pi}{12}$

A19 The sequence generated by the recurrence $a_{n+1} = \sqrt{2+a_n}$ for positive integers n and $a_1 = 1$ is known to be convergent. What is $\lim_{n\to\infty} a_n$?

A. -1 **B.** 1 **C.** $\sqrt{3}$ **D.** 2

A20 The infinite series $1 - 2x + 2^2x^2 - 2^3x^3 + \ldots$ has radius of convergence

A. 1 **B.** 2 **C.** -2 **D.** $\frac{1}{2}$

A21 What is the value of k which makes the function

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq 1 \\ 20x - 18, & \text{if } x > 1 \end{cases}$$

continuous at $x = 1$?

A. 1 **B.** 2 **C.** -1 **D.** -2

A22 $\dfrac{d}{dx} \displaystyle\int_0^x \cos(t^{100})\,dt =$

A. $x^{101} \sin(x^{100})$ **B.** $\dfrac{x^{101} \sin x}{101}$ **C.** $\cos(x^{101})$ **D.** $\cos(x^{100})$

A23 The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$ with derivative $f'(0) = 1$. If $f(0) = 1$ then which of the following best approximates $f(h)$ for small h?

A. 1 **B.** $1 + h$ **C.** $1 - h$ **D.** h^2

A24 If $\displaystyle\lim_{x \to 0} \frac{f(x)}{x} = 1$ then $\displaystyle\lim_{x \to 0} \frac{x}{f(5x)} =$

A. $-\dfrac{1}{5}$ **B.** $\dfrac{1}{5}$ **C.** 1 **D.** depends on f

A25 If $\displaystyle\int_0^4 f(x)\,dx = 2$ then $\displaystyle\int_0^2 x f(x^2)\,dx =$

A. 0 **B.** 1 **C.** 2 **D.** 3

A26 An exam paper has 13 questions out of which 10 are to be attempted including at least 4 from the first 5 questions. What is the number of choices available?

A. 56 **B.** 140 **C.** 196 **D.** 280

Section B:

Full credit will only be awarded to clearly presented and logically coherent solutions.

B1 Use induction to prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

for all $n \in \mathbb{N}$. ***(4 marks)***

B2 Find the general solution to the differential equation

$$\frac{d^2y}{dt^2}-5\frac{dy}{dt}+6y=e^t$$

where y is a function of t. ***(4 marks)***

B3 Sketch the following three sets of complex numbers on an Argand diagram:

$$A := \{z \in \mathbb{C} \ : \ \mathrm{Re}((1-i)z)=2\};$$
$$B := \{z \in \mathbb{C} \ : \ |z-2i|=2\};$$
$$C := \left\{z \in \mathbb{C} \ : \ 0 \le \mathrm{Arg}(z) \le \frac{\pi}{2}\right\}.$$

Also, determine the sets $A \cap B$ and $A \cap B \cap C$. ***(4 marks)***

B4 Sketch a picture to illustrate why the estimates

$$(b-a)\sqrt{a} \le \int_a^b \sqrt{x}\,dx \le (b-a)\sqrt{b}$$

hold for $0 \le a < b$.

Now let $0 < r < 1$ be given. Obtain the estimates

$$\frac{(1-r^2)r}{1-r^3} \le \int_0^1 \sqrt{x}\,dx \le \frac{1-r^2}{1-r^3}$$

by using the points $\cdots < r^{2(n+1)} < r^{2n} < \cdots < r^4 < r^2 < 1$ to divide the interval $[0,1]$ into (an infinite number of) subintervals. Deduce that

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}.$$

(6 marks)

B5 Write down the Maclaurin series expansion of $\frac{1}{1-x}$, and use it to show that

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n.$$

Verify that

$$\frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} + 2\left(\frac{(k-1)k}{2}\right) = k^2 + 1,$$

and determine the Maclaurin series expansion of $\frac{1-x+2x^2}{(1-x)^3}$. Hence evaluate

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \dots.$$

You may assume that all Maclaurin series involved converge whenever $|x| < 1$.

(6 marks)

B6 Let $f(1) \geq f(2) \geq f(3) \geq \cdots \geq f(n) \geq f(n+1) \cdots \geq 0$ be a decreasing sequence of non-negative real numbers. Explain briefly why

$$f(2^n) + f(2^n + 1) + \cdots + f(2^{n+1} - 1) \leq 2^n f(2^n)$$

for $n = 1, 2, \dots$.

By taking $f(k) = 1/k^3$ and considering the $(2^{n+1} - 1)$-th partial sum or otherwise, show that the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges to a limit no bigger than $\frac{4}{3}$.

(5 marks)

End of Question Paper