SCHOOL OF MATHEMATICS AND STATISTICS

Mathematics Core 1

2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: 55

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student
Section A:

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. (26 marks)

A1 Let \( A = \{1, 2\} \) and \( B = \{2, 3\} \). Which of the following options is an element of \( A \times B \)?

A. \((2, 1)\)  
B. \(\{2, 2\}\)  
C. \(\{(2, 2)\}\)  
D. \((2, 2)\)

A2 The number of subsets of \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) is

A. \(2^{10}\)  
B. \(10^2\)  
C. \(2^5\)  
D. \((2 \times 10)^{10}\)

A3 How many different functions from \( \{x, y, z\} \) to \( \{a, b\} \) are there?

A. 6  
B. 8  
C. 9  
D. \(\binom{3}{2}\)

A4 How many different anagrams does \(\text{ABCDEABCDE}\) have?

A. \(\frac{10!}{2^5}\)  
B. \(\frac{10!}{5!}\)  
C. \(\frac{10!}{5^2}\)  
D. \(\frac{10^{10}}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}\)

A5 If \( r \neq 1 \) then \( r^2 + r^4 + r^6 + r^8 + r^{10} \) is equal to

A. \(\frac{r^2(1 + r^{10})}{1 + r^2}\)  
B. \(\frac{r^2(1 - r^{10})}{1 - r^2}\)  
C. \(\frac{r^2(1 + r^2)^5}{1 + r^5}\)  
D. \(\frac{1 - r^{10}}{1 - r^2}\)

A6 If \( 0 < \theta < \frac{\pi}{2} \) and \( \cos \theta = \frac{3}{5} \) then \( \sin(2\theta) = \)

A. \(\frac{6}{25}\)  
B. \(\frac{12}{25}\)  
C. \(\frac{18}{25}\)  
D. \(\frac{24}{25}\)
A7  If \( \tan x = \frac{1}{2} \) and \( \tan y = \frac{1}{3} \) then \( \tan(x + y) = \)

A. 0  B. 1  C. 2  D. 3

A8 \( \lim_{x \to 0} \frac{e^{2x} - 1}{x} = \)

A. 2  B. 1  C. 0  D. \( \infty \)

A9 The function \( f : \mathbb{R} \to \mathbb{R} \) has the property that \( x^2 \leq f(x) \leq x^2 + 1 \) for \( x > 0 \). Then \( \lim_{x \to \infty} \frac{f(x)}{x^2} = \)

A. 0  B. 1  C. 2  D. \( \infty \)

A10 If \( e^y = \sin x \) for \( 0 < x < \pi \) then \( \frac{dy}{dx} = \)

A. \( \tan x \)  B. \( \cot x \)  C. \( -\tan x \)  D. \( 3 - \cot x \)

A11 The tangent to the curve \( x^3 + y^3 = 2 \) at the point \((1, 1)\) has slope

A. 1  B. 0  C. \(-1\)  D. \( \infty \)

A12 \( \int_0^2 \frac{x^2}{1 + x^3} \, dx = \)

A. 0  B. \( \frac{2}{3} \)  C. \( \ln(3) \)  D. \( \frac{2}{3} \ln(3) \)

A13 \( 1 + \int_1^2 \ln(x) \, dx = \)

A. 2  B. \( \ln(2) \)  C. \( \ln(3) \)  D. \( 2\ln(2) \)
A14 \[
\int_0^\infty \frac{1}{1+x^2} \, dx =
\]
A. \( \pi \) \hspace{1cm} B. \( \frac{\pi}{2} \) \hspace{1cm} C. \( \frac{\pi}{3} \) \hspace{1cm} D. \( \frac{\pi}{4} \)

A15 \( \frac{1-3i}{1+2i} = \)
A. \(-1-i\) \hspace{1cm} B. \(5-i\) \hspace{1cm} C. \(-1+i\) \hspace{1cm} D. \(1+i\)

A16 If \( \frac{\pi}{2} < \theta < \pi \) and \( \sin \theta = \frac{3}{5} \) then \( e^{i\theta} = \)
A. \( \frac{4+3i}{5} \) \hspace{1cm} B. \( \frac{4-3i}{5} \) \hspace{1cm} C. \( \frac{-4+3i}{5} \) \hspace{1cm} D. \( \frac{-4-3i}{5} \)

A17 If \( z = 2+i \) then \( |z^4| + |\overline{z^4}| = \)
A. \(25\) \hspace{1cm} B. \(50\) \hspace{1cm} C. \(1000\) \hspace{1cm} D. \(1250\)

A18 If \( \arg(z) = \frac{\pi}{3} \) and \( \arg(w) = \frac{\pi}{4} \) then \( \arg \left( \frac{z}{w} \right) = \)
A. \( \frac{7\pi}{12} \) \hspace{1cm} B. \( \frac{\pi}{5} \) \hspace{1cm} C. \( \frac{4}{3} \) \hspace{1cm} D. \( \frac{\pi}{12} \)

A19 The sequence generated by the recurrence \( a_{n+1} = \sqrt{2} + a_n \) for positive integers \( n \)
and \( a_1 = 1 \) is known to be convergent. What is \( \lim_{n \to \infty} a_n ? \)
A. \(-1\) \hspace{1cm} B. \(1\) \hspace{1cm} C. \(\sqrt{3}\) \hspace{1cm} D. \(2\)

A20 The infinite series \( 1 - 2x + 2^2 x^2 - 2^3 x^3 + \ldots \) has radius of convergence
A. \(1\) \hspace{1cm} B. \(2\) \hspace{1cm} C. \(-2\) \hspace{1cm} D. \(\frac{1}{2}\)
A21 What is the value of $k$ which makes the function

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq 1 \\ 20x - 18, & \text{if } x > 1 \end{cases}$$

continuous at $x = 1$?

A. 1  B. 2  C. −1  D. −2

A22 \[
\frac{d}{dx} \int_0^x \cos(t^{100}) \, dt =
\]

A. $x^{101} \sin(x^{100})$  B. $\frac{x^{101} \sin x}{101}$  C. $\cos(x^{101})$  D. $\cos(x^{100})$

A23 The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $x = 0$ with derivative $f'(0) = 1$. If $f(0) = 1$ then which of the following best approximates $f(h)$ for small $h$?

A. 1  B. $1 + h$  C. $1 - h$  D. $h^2$

A24 If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ then $\lim_{x \to 0} \frac{x}{f(5x)} =

A. $-\frac{1}{5}$  B. $\frac{1}{5}$  C. 1  D. depends on $f$

A25 If $\int_0^4 f(x) \, dx = 2$ then $\int_0^2 x f(x^2) \, dx =

A. 0  B. 1  C. 2  D. 3

A26 An exam paper has 13 questions out of which 10 are to be attempted including at least 4 from the first 5 questions. What is the number of choices available?

A. 56  B. 140  C. 196  D. 280
Section B:

Full credit will only be awarded to clearly presented and logically coherent solutions.

B1 Use induction to prove that

\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}
\]

for all \( n \in \mathbb{N} \). \hfill (4 marks)

B2 Find the general solution to the differential equation

\[
\frac{d^2y}{dt^2} - 5 \frac{dy}{dt} + 6y = e^t
\]

where \( y \) is a function of \( t \). \hfill (4 marks)

B3 Sketch the following three sets of complex numbers on an Argand diagram:

\[
A := \{ z \in \mathbb{C} : \text{Re}((1 - i)z) = 2 \};
\]

\[
B := \{ z \in \mathbb{C} : |z - 2i| = 2 \};
\]

\[
C := \left\{ z \in \mathbb{C} : 0 \leq \text{Arg}(z) \leq \frac{\pi}{2} \right\}.
\]

Also, determine the sets \( A \cap B \) and \( A \cap B \cap C \). \hfill (4 marks)

B4 Sketch a picture to illustrate why the estimates

\[
(b - a)\sqrt{a} \leq \int_a^b \sqrt{x} \, dx \leq (b - a)\sqrt{b}
\]

hold for \( 0 \leq a < b \).

Now let \( 0 < r < 1 \) be given. Obtain the estimates

\[
\frac{(1 - r^2)r}{1 - r^3} \leq \int_0^1 \sqrt{x} \, dx \leq \frac{1 - r^2}{1 - r^3}
\]

by using the points \( \cdots < r^{2(n+1)} < r^{2n} < \cdots < r^4 < r^2 < 1 \) to divide the interval \( [0, 1] \) into (an infinite number of) subintervals. Deduce that

\[
\int_0^1 \sqrt{x} \, dx = \frac{2}{3}.
\] \hfill (6 marks)
B5 Write down the Maclaurin series expansion of \( \frac{1}{1-x} \), and use it to show that
\[
\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n.
\]
Verify that
\[
\frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} + 2 \left( \frac{(k-1)k}{2} \right) = k^2 + 1,
\]
and determine the Maclaurin series expansion of \( \frac{1-x+2x^2}{(1-x)^3} \). Hence evaluate
\[
1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \ldots.
\]
You may assume that all Maclaurin series involved converge whenever \(|x| < 1\).

(6 marks)

B6 Let \( f(1) \geq f(2) \geq f(3) \geq \cdots \geq f(n) \geq f(n+1) \cdots \geq 0 \) be a decreasing sequence of non-negative real numbers. Explain briefly why
\[
f(2^n) + f(2^n + 1) + \cdots + f(2^{n+1} - 1) \leq 2^n f(2^n)
\]
for \( n = 1, 2, \ldots \).

By taking \( f(k) = \frac{1}{k^3} \) and considering the \((2^{n+1} - 1)\)-th partial sum or otherwise, show that the series \( \sum_{k=1}^{\infty} \frac{1}{k^3} \) converges to a limit no bigger than \( \frac{4}{3} \).

(5 marks)

End of Question Paper