The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Advanced Calculus and Linear Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 100 marks.

1 Compute the integral $\iiint_D xz\,dx\,dy\,dz$, where D is the region described by the inequalities $0 \leq z \leq 2$, $x \geq 0$, $y \geq 0$, $x + y \leq 1$. ***(5 marks)***

2 (i) Compute the volume of the parallelopiped with edges $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}2\\0\\2\end{pmatrix}$, $\begin{pmatrix}1\\3\\0\end{pmatrix}$. ***(2 marks)***

(ii) Compute the volume of the parallelopiped with edges

$$\begin{pmatrix}-3/4\\3/4\\1/4\end{pmatrix}, \begin{pmatrix}1/4\\-1/4\\1/4\end{pmatrix}, \begin{pmatrix}3/4\\-1/4\\-1/4\end{pmatrix}.$$

(3 marks)

(iii) What do you observe, and can you explain why this happened? [Hint: try multiplying two matrices together.] ***(2 marks)***

3 Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(r, \theta, \phi) = (x, y, z) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

associated to spherical polar coordinates.

You may assume that its Jacobian determinant, $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = -r^2 \sin\phi$.

You may also assume without justification that the volume of a spherical ball of radius R is $\dfrac{4\pi}{3} R^3$.

Compute the average value of the function ϕ on the upper half H of a spherical ball B of radius R centred on the origin. ***(7 marks)***

4 Let $A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & -1 \\ 2 & 3 & 1 & 1 \end{pmatrix}$, and consider the linear map $\ell_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $\ell_A(\mathbf{x}) = A\mathbf{x}$.

(i) By row reduction to reduced echelon form (i.e. complete Gaussian elimination), find a basis for $\ker(\ell_A)$, the null-space of A. Determine also the rank of ℓ_A. ***(5 marks)***

(ii) Find a subset of the columns of A that is a basis for $\mathrm{im}(\ell_A)$, and express each of the other columns as a linear combination of these basis elements. ***(4 marks)***

(iii) Give an implicit description of the subspace $\mathrm{im}(\ell_A)$ of $\mathbb{R}^3$, i.e. a homogeneous linear equation (or system of equations) in x, y, z, characterising those $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathrm{im}(\ell_A)$. (You may be able to just spot one by inspection.) ***(2 marks)***

(iv) Write down a matrix $B \in M_{4,2}(\mathbb{R})$ such that $\ker(\ell_A) = \mathrm{im}(\ell_B)$, and a matrix $C \in M_{1,3}(\mathbb{R})$ such that $\mathrm{im}(\ell_A) = \ker(\ell_C)$. ***(2 marks)***

5 (i) Define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x^2 + y^2 - z^2 - 1$. Compute the derivative matrix $D(F)$, i.e the gradient grad F.
Use it to find an equation for the tangent plane to the surface $S : x^2 + y^2 - z^2 = 1$ at the point $P = (1, 1, 1)$. ***(4 marks)***

(ii) Consider the map $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{r}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} \cosh u \cos v \\ \cosh u \sin v \\ \sinh u \end{pmatrix}$.
(Recall that $\cosh u := \dfrac{e^u + e^{-u}}{2}$ and $\sinh u := \dfrac{e^u - e^{-u}}{2}$. You may assume standard properties of these hyperbolic functions.)
Show that any $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the form $\mathbf{r}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$ satisfies the equation for S from part (i). What is the composite map $F \circ \mathbf{r} : \mathbb{R}^2 \to \mathbb{R}$? ***(4 marks)***

(iii) Compute the derivative matrix $D(\mathbf{r})\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$.
Calculate directly the product $D(F)\left(\mathbf{r}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)\right) D(\mathbf{r})\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$, and check that it is the same as $D(F \circ \mathbf{r})\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$. ***(7 marks)***

(iv) Find a $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$ such that $\mathbf{r}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$ gives the point P. ***(3 marks)***

6 (i) Find the stationary points for the function $f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$ and classify them. ***(9 marks)***

(ii) Write down the definition of the gradient of a general, smooth function $g(x, y, z)$. Show that curl grad $g = 0$ (i.e. show that $\nabla \times \nabla g = 0$). ***(3 marks)***

(iii) Find the curl of the vector field $\mathbf{F} = (2xz, 2yz^2, x^2 + 2y^2 z - 1)$. ***(3 marks)***

(iv) Find the function ϕ for which $\mathbf{F} = \text{grad } \phi$, with $\mathbf{F}$ given above. ***(5 marks)***

(v) Let $\Psi(x, y, z) = x^3 + xz + yz + 3$. Find the directional derivative of Ψ at the point $(1, 2, 3)^{\mathrm{T}}$ in the direction of $\mathbf{r} = (1, 2, 0)^{\mathrm{T}}$. ***(5 marks)***

7 (i) State the Divergence Theorem. ***(3 marks)***

(ii) In this part, you will show that the Divergence Theorem holds for a specific case.

(a) Evaluate the surface integral $\int_{\mathcal{S}} \mathbf{r}\cdot \mathrm{d}\mathbf{S}$ directly, where $\mathbf{r}$ is the position vector, over the surface $\mathcal{S}$ given by $z = 1 - x^2 - y^2$ for which $z \geq 0$. To perform the calculation, use the following parametrisation for x, y and z:

$$x = \sin u \cos v,\;\; y = \sin u \sin v,\;\; z = \cos^2 u$$

(with $0 \leq u \leq \pi/2$ and $0 \leq v \leq 2\pi$), and show that

$$\mathbf{r} \cdot \mathrm{d}\mathbf{S} = (2\sin^3 u \cos u + \cos^3 u \sin u)\mathrm{d}u\mathrm{d}v.$$

You are then able to perform the integral over u and v.

(13 marks)

(b) What is div $\mathbf{r}$? Using appropriate coordinates, perform the volume integral $\int_V \mathrm{div}\ \mathbf{r}\ \mathrm{d}V$, where V is the volume enclosed by the surface $\mathcal{S}$, and show that both sides of the Divergence Theorem are equal.

(9 marks)

End of Question Paper