Data provided: formula sheet

The University Of Sheffield.



**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2019–20**

**Mathematics II (Electrical)** **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** (i) (a) Let $f(t)$ be the function defined for $t \geq 0$ by

$$f(t) \ := H(t-1)t - t^2e^t - 2,$$

where $H$ is the Heaviside function. Calculate the Laplace transforms of $f(t)$. **(4 marks)**

(b) Consider the differential equation

$$y''(t) + 3y'(t) + 2y(t) \ = 5e^t$$

with initial conditions $y(0) = y'(0) = 0$. Show that the Laplace transform of $y(t)$ is given by

$$Y(s) \ = \ \frac{5}{(s-1)(s+2)(s+1)}.$$

**(4 marks)**

(c) Hence, or otherwise, find the solution to

$$y''(t) + 3y'(t) + 2y(t) \ = 5e^t$$

with initial conditions $y(0) = y'(0) = 0$. **(8 marks)**

(ii) Derive the expression for the frequency shift of a Laplace Transform, as given in the Formula Sheet. **(4 marks)**

**2** Consider the function $f(x) = x$ over the region $[0, \pi]$.

(i) Show that the Fourier series, $S[f](x)$, over the period $[-\pi, \pi]$ is

$$S[f](x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2} \cos((2n-1)x)$$

when an **even** extension is applied. **(12 marks)**

(ii) Sketch the graph of the Fourier series you calculated in Part (a) over the interval $[-2\pi, 3\pi]$. **(3 marks)**

(iii) By using your answer to part (a) and a careful choice of $x$, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

**(5 marks)**

**3** (i) Let $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$. Calculate the partial derivatives

$$f_x,\ f_y,\ f_{xx},\ f_{yx},\ f_{xy}.$$

**(5 marks)**

(ii) Hence, find and classify *all* the critical points of the function $f(x, y)$.

**(12 marks)**

(iii) Let $p(x, y)$ be a function with $p(0, y) = y^{-1}$, $p(x, 1) = e^x + e^{2x}$ and $p_{yx} = xe^{xy}$. Find $p(x, y)$. **(3 marks)**

**4** (i) Let $R = \{(x, y) \mid 0 \le x \le \pi,\ 1 \le y \le 2\}$. Calculate

$$\iint_R x\cos(xy)\ dA.$$

**(7 marks)**

(ii) Consider

$$\int_0^8 \int_{y^{1/3}}^2 f(x, y) dx dy.$$

for some $f(x, y)$.
Sketch the region of integration and, hence, change the order of integration.
(Note: You do not need to solve the integral) **(5 marks)**

(iii) Calculate the surface area of a curve $f(x, y) = x^2 + 2y$, in the region $R = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Hint: Let $\sinh(u) = 2x/\sqrt{5}$. **(8 marks)**

**5** (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = \left(\cos(xyz),\ \frac{x}{yz},\ x^2 + y^2 + z^2\right).$$

(a) Calculate **div F**. **(3 marks)**

(b) Calculate **curl F**. **(3 marks)**

(c) Calculate the Laplacian of F. **(3 marks)**

(d) Calculate **div**( **curl F**) **(3 marks)**

(ii) Show that

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F},$$

for some vector field **F**. **(8 marks)**

## End of Question Paper

# MAS241 FORMULA SHEET

## Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)dt.$$

**Properties of the Laplace transform:** $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

| | |
|---|---|
| $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ | linearity |
| $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ | differentiation w.r.t. $t$ |
| $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ | second differentiation w.r.t. $t$ |
| $\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$ | frequency shift |
| $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad (\text{for } a > 0)$ | time shift |
| $\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad (\text{for } a > 0)$ | scaling |
| $\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal) | convolution |

**Table of standard Laplace transforms:**

| $f(t)$ | $\mathcal{L}\{f(t)\}(s)$ | Region of validity |
|---|---|---|
| $t^n \qquad (\text{for } n \geq 0)$ | $\frac{n!}{s^{n+1}}$ | $Re(s) > 0$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ | $Re(s) > 0$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ | $Re(s) > 0$ |
| $H(t-T) \qquad (\text{for } T \geq 0)$ | $\frac{e^{-sT}}{s}$ | $Re(s) > 0$ |
| $\delta(t-T) \qquad (\text{for } T \geq 0)$ | $e^{-sT}$ | $s \in \mathbb{C}$ |

## Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$

**Properties of the Fourier transform:** $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

| | |
|---|---|
| $\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$ | frequency shift |
| $\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$ | time shift |
| $\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$ | differentiation |
| $\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$ | symmetry |
| $\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$ | scaling |
| $\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$ | convolution |

**Table of standard Fourier transforms:**

| $f(t)$ | $\mathcal{F}\{f(t)\}(\omega)$ |
|---|---|
| $e^{-a\lvert t\rvert}$ (for $a > 0$) | $\frac{2a}{a^2+\omega^2}$ |
| $\mathrm{rect}_T(t)$ | $\mathrm{sinc}(\frac{T\omega}{2})$ |
| 1 | $2\pi\delta(\omega)$ |

## Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period $T$ is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

## Coordinate systems:

### Cylindrical polar coordinates

$$(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r,\theta,z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\Big)$$
$$dV = r dr d\theta dz.$$

### Spherical polar coordinates

$$(x,y,z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho,\theta,\phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\Big)$$
$$dV = \rho^2\sin(\phi)d\rho d\phi d\theta.$$