Data provided: Formula sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.
The paper is marked out of a total of 60 marks.

1 (i) (a) A continuous random variable X has probability density function

$$p(x) = \begin{cases} cx^2 & \text{for} \quad 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of the constant c.
Hence calculate $P(1 \leq X \leq 2)$, where P denotes the probability.
(1 mark)

(b) The probability density function, $u(r)$, of a continuous random variable R has a uniform distribution given by

$$u(r) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{for} \quad \alpha \leq r \leq \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where α and β are positive constants such that $\beta \geq \alpha$.
Find the mean and variance of this uniform distribution.
(2 marks)

Find the probability that $R \leq \alpha + h(\beta - \alpha)$, where $0 \leq h \leq 1$.
(2 marks)

(c) A continuous random variable Y satisfies the normal distribution and has probability density function

$$f(y) = \frac{1}{7\sqrt{2\pi}} \exp\left(-\frac{(y+2)^2}{98}\right), \quad \text{for } -\infty < y < \infty.$$

What are its mean and variance? ***(2 marks)***

(ii) (a) Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F} = (x^2 y, xz^3, -y^2 z^2).$$

Calculate $\nabla^2 \mathbf{F}$. ***(2 marks)***

(b) The scalar field V is defined by

$$V = x^2 y^2 z^2.$$

Calculate $\nabla^2 V$. ***(2 marks)***

(c) Find the values of the constants a and b for which the vector field

$$\mathbf{G} = (xyz)^b (x^a \mathbf{i} + y^a \mathbf{j} + z^a \mathbf{k})$$

is an irrotational vector. ***(4 marks)***

2 In the context of a vibrating string, describe briefly what is meant by a "normal mode". ***(1 mark)***

Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

has solutions of the form $y(x, t) = f(x + \lambda t)$ for arbitrary twice differentiable functions f, provided that $\lambda = -1$ or $\lambda = 1$. ***(4 marks)***

Write down the general solution of the partial differential equation. ***(1 mark)***

Find the particular solution for y that satisfies the conditions

$$y(x, 0) = \sin 2x$$

and

$$\frac{\partial y}{\partial t}(x, 0) = \sin x.$$

(9 marks)

3 (i) The power, $P = P(E, R)$, consumed in an electrical resistor is given by

$$P = \frac{E^2}{R},$$

where E is the voltage through the resistor (in volts), R is the resistance (in ohms) and P is the power in watts. Suppose $E = 200$ V and $R = 8\ \Omega$. Then $P = 5000\ W$. Use the formula for small increments to

(a) estimate the change in P if E is increased by 5 V and R is decreased by 0.4 Ω,

(b) estimate the percentage change in P if E is increased by 0.3% and R is decreased by 0.2%. ***(7 marks)***

(ii) Verify that the equation

$$2\frac{\partial^2 u}{\partial x^2} + y^3 \frac{\partial^2 u}{\partial y \partial x} = 0$$

is satisfied by the function $u(x, y) = \ln(1 + xy^2)$. ***(4 marks)***

(iii) Write down the iteration formula for the Newton-Raphson method. Starting with an initial guess of $x_0 = 0.5$, use the Newton-Raphson method to find a root of the function $f(x) = \cos x - 2x$ correct to 4 decimal places. ***(4 marks)***

4 (i) Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $x|x|$

(b) $\exp(-|x|)$

(c) $\pi - |x|$ ***(3 marks)***

(ii) A periodic function, $f(t)$, with period 2π is defined by

$$f(t) = \pi - |t| \quad \text{for} \quad -\pi \le t < \pi.$$

(a) Sketch a graph of the function $f(t)$ for values of t from $t = -3\pi$ to $t = 3\pi$. ***(2 marks)***

(b) Calculate the first four non-zero terms of the Fourier series for $f(t)$. ***(6 marks)***

(iii) The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $h(x)$ to be equal to the sum of its Fourier series at each point where it is continuous. State two of the Dirichlet conditions. ***(2 marks)***

For each of the following functions state whether or not the Dirichlet conditions are satisfied. For those cases where the Dirichlet conditions are not satisfied, give a brief explanation of why that is the case.

(a) A periodic function, $g(x)$, of period 2π, defined by

$$g(x) = \frac{1}{25 - x^2} \quad \text{for} \quad -\pi \le x < \pi.$$

(b) A periodic function, $p(x)$, of period 2π, defined by

$$p(x) = \sin\left[\frac{1}{x+2}\right] \quad \text{for} \quad -\pi \le x < \pi.$$

(2 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$