



The
University
Of
Sheffield.

MAS250

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2018–19

Mathematics II (Materials)

2 hours

Marks will be awarded for answers to all questions in Section A, and for your best THREE answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.

The maximum possible mark for the paper is 100.

Section A

A1 Find the particular solution of the equation

$$\frac{dy}{dx} + y \cos x = 2x e^{-\sin x}$$

which satisfies $y = 0$ when $x = 1$.

(8 marks)

A2 Find the general solution of the equation

$$4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + y = e^x.$$

(9 marks)

A3 The pressure (p), volume (V) and temperature (T) of a container full of gas are related by the ideal gas law

$$pV = nRT,$$

where n is a constant which measures the number of gas particles, and R is the universal gas constant.

The pressure is initially 10^5 Nm^{-2} . Using the chain rule, estimate the pressure if the volume increases by 2% and the temperature increases by 3% *(7 marks)*

A4 Two quantities x and y have means 3.4 and -4.56 respectively, standard deviations 1.39 and 2.52 respectively, and covariance -3.48.

(a) Calculate the correlation coefficient between x and y , correct to 3 significant figures. *(2 marks)*

(b) It is assumed that x and y satisfy the linear relationship

$$y = a + b(x - \bar{x}), \quad (*)$$

where \bar{x} is the mean of x .

Calculate the least squares estimates of a and b , correct to 3 significant figures. State, giving a reason, whether you expect (*) to give a good model. *(4 marks)*

A5 Find a vector normal to the surface $\phi = -1$ at the point A with coordinates $(1, 2, 0)$, where

$$\phi = y \sin \pi z + x^2 y z - 2x \cos \pi y + x \cosh z. \quad (6 \text{ marks})$$

Find also the directional derivative of ϕ at A , in the direction $\mathbf{d} = (2, 0, -1)$. *(4 marks)*

Section B

B1 (a) Find the general solution of the equation

$$y \frac{dy}{dx} - y^2 = 1. \quad (7 \text{ marks})$$

(b) Find the solution of the equation

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = e^{2x}$$

given that $y = 1$ and $dy/dx = 0$ when $x = 0$. *(13 marks)*

- B2 (a) The following table shows the MAS110 marks (= y) and the UCAS points scores (= x) of 10 students:

x	184	160	136	160	144	128	124	128	200	142
y	76	69	49	58	56	68	50	40	86	68

Showing your working, calculate the mean and standard deviation of x and of y , and also the correlation between x and y . Give all your answers correct to three significant figures. **(11 marks)**

Comment briefly on the implications of the correlation between x and y . **(1 mark)**

- (b) A vector field \mathbf{u} is given by

$$\mathbf{u} = (\cosh(xy) + z^2, xyz - ye^x, y \sin z - z \cos x).$$

Evaluate

$$\nabla \cdot (\nabla \times \mathbf{u}). \quad \textbf{(8 marks)}$$

- B3 (a) By integrating by parts twice, show that

$$\int_0^1 e^x \cos n\pi x \, dx = \frac{(-1)^n e - 1}{1 + n^2 \pi^2},$$

where n is a positive integer. **(9 marks)**

- (b) A function $f(x) = e^x$ is defined on the interval $0 \leq x \leq 1$.

- (i) Show that $f(x)$ can be represented by the Fourier cosine series

$$e^{-1} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e - 1}{1 + n^2 \pi^2} \cos n\pi x. \quad \textbf{(8 marks)}$$

- (ii) Sketch the function given by the above Fourier cosine series on the interval $-2 \leq x \leq 2$. **(3 marks)**

- B4** A string of unstretched length $l > 0$ has its ends fixed at $x = 0$ and $x = l$. Its displacement $y(x, t)$ in the transverse direction satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where c is a positive constant.

- (a) Using separation of variables $y(x, t) = X(x)T(t)$, show that

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \alpha,$$

where α is a constant.

Given that $\alpha < 0$, deduce that the general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[D_n \cos\left(\frac{n\pi ct}{l}\right) + E_n \sin\left(\frac{n\pi ct}{l}\right) \right],$$

where D_n and E_n are constants. **(15 marks)**

- (b) If $y = 0$ and $\frac{\partial y}{\partial t} = 3c \sin\left(\frac{3\pi x}{l}\right)$ at $t = 0$, show that

$$y(x, t) = \frac{l}{\pi} \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi ct}{l}\right). \quad \text{(5 marks)}$$

End of Question Paper

FORMULA SHEET

Trigonometry

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$a \cos \theta + b \sin \theta = R \cos(\theta - \alpha), \text{ where } R = \sqrt{(a^2 + b^2)}, \cos \alpha = a/R \text{ and } \sin \alpha = b/R$$

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1$$

$$\sinh^{-1} x = \ln \left[x + \sqrt{(1 + x^2)} \right], \quad \text{all } x$$

$$\cosh^{-1} x = \ln \left[x + \sqrt{(x^2 - 1)} \right], \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right), \quad |x| < 1$$

$$\operatorname{coth}^{-1} x = \frac{1}{2} \ln \left(\frac{x + 1}{x - 1} \right), \quad |x| > 1$$

Differentiation and Integration

Function	Derivative
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{coth} x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \operatorname{coth} x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad x < 1$
$\operatorname{coth}^{-1} x$	$-\frac{1}{x^2-1}, \quad x > 1$

Function	Integral
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1} \left(\frac{x}{a} \right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\int_a^b uv dx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v) dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t , i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s , i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

First-Order Differential Equations**1. Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x) dx + C$$

2. Separation of Variables

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x) dx$$

3. Homogeneous Equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x \frac{dz}{dx} = f(z)$$

4. Linear Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x) dx}$ to give

$$\frac{d}{dx} \left(y e^{\int P(x) dx} \right) = Q(x) e^{\int P(x) dx}$$

The Second-Order Differential Equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

where a , b , and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c , is given by

- (i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,
- (ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal ($m_1 = m_2 = m$),
- (iii) $y_c = e^{px}(A \cos qx + B \sin qx)$, if m_1 and m_2 are complex ($m_1 = p + iq$, $m_2 = p - iq$), where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$$f(x) = Ax^2 + Bx + C \quad y_p = ax^2 + bx + c$$

$$f(x) = Ae^{kx} \quad y_p = ae^{kx}$$

when k is not one of the roots of the auxiliary equation

$$f(x) = Ae^{kx} \quad y_p = axe^{kx}$$

when k is one of the roots of the auxiliary equation

$$f(x) = A \cos mx + B \sin mx \quad y_p = a \cos mx + b \sin mx$$

when $\sin mx$ or $\cos mx$ is not part of the complementary function

$$f(x) = A \cos mx + B \sin mx \quad y_p = x(a \cos mx + b \sin mx)$$

when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leq x \leq l$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

where

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots$$

On the interval $0 \leq x \leq l$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

$$\text{Means } \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{etc.}$$

$$\text{Variances } s_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^n (x_i^2) - \bar{x}^2 \quad \text{etc.}$$

s_x is standard deviation

$$\text{Covariance } \text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n} \sum_{i=1}^n (x_i y_i) - \bar{x} \bar{y}$$

$$\text{Correlation coefficient } r = \frac{\text{cov}(x, y)}{s_x s_y}$$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \quad b = \frac{\text{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.