Data provided: Formula sheet

MAS252

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The vertical oscillations of a thin flexible metallic rod of length L are given by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2}; \quad u = u(x,t),$$

where x denotes the position along the rod. These oscillations are subject to the boundary conditions

$$u(0,t) = u(L,t) = 0,$$

and the initial conditions

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0; \quad u(x,0) = x(L-x).$$

Use the method of separation of variables to determine the solution, $u(x,t)$, of the above wave equation. ***(25 marks)***

2 (i) The equation

$$4xe^{2x} - 2 = 0,$$

is expected to have a root near $x_0 = 0.5$. Perform *four* iterations of the Newton-Raphson method to find this root. Show your calculations correct to *three* decimal places. ***(7 marks)***

 Question 2 continued on next page

2 (continued)

(ii) Find the first three non-zero terms of the Taylor series solution of the differential equation

$$y'' - \frac{4x}{x^2+1}y' + \frac{6}{x^2+1}y = 0; \quad y = y(x),$$

subject to the conditions $y(1) = 1$ and $y'(1) = 0$. Hence, use the series representation to calculate $y(1.1)$. Give your answer correct to *four* decimal places. ***(12 marks)***

(iii) If $w = 1/r$ and $r = x^2 + y^2 + z^2$, use the chain rule to show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -\frac{2}{r}.$$

(6 marks)

3 (i) Find the Fourier series representation of the function

$$f(x) = \begin{cases} (x+\pi)^2, & \text{for } -\pi \le x \le 0 \\ (x-\pi)^2, & \text{for } \quad 0 \le x \le \pi \end{cases}$$

in the interval $-\pi \le x \le \pi$. Use the determined Fourier series representation of $f(x)$ at $x = \pi$ to prove the identity

$$\pi^2 = 12\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots\right).$$

(17 marks)

(ii) In the process of discretisation the second order derivates (written as finite differences) can be written as

$$\frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i.j} + U_{i-1,j}}{h^2}$$

where $U = U(x, y)$, $U_{i.j} = U(ih, jk)$ and h is the step-size used in the discretisation along the x-axis. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2$$

is satisfied over the square domain $-1 \le x \le 1$, $-1 \le y \le 1$, with $h = k = 0.5$ and it is subject to the boundary conditions $u(x, y) = 0$ at the boundaries of the domain that correspond to $x = \pm 1$ and $y = \pm 1$. Using a finite difference approximation, represent graphically the domain of interest (with the grid points), and derive the equations required to find $u(x, y)$ at every node of the grid taking into account possible symmetries. **Do not attempt** to find the solutions of these equations. ***(8 marks)***

4 The profit of a company producing chairs is given by the function

$$f(x, y, z) = \frac{2xy}{\sqrt{z}}$$

where the variables x, y, and z denote the units of raw material needed, the units of labour needed, and the storage space of finished products, respectively. The maximum profit of the company is obtained when $x = 40$, $y = 23$, and $z = 1.5$. Let us suppose that, compared to the values corresponding to the maximum profit, the available raw material (x) is reduced by 10% and the storage capacity (z) is increased by 15%. Use the small increment formula (or the small error formula) to calculate by how much the units of labour (y) need to be changed so that the profit of the company is unchanged. ***(9 marks)***

(i) Let is consider the function

$$f(x, y) = e^{2x} \cos(y - x).$$

(a) Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} = 0$$

(10 marks)

(b) Let us write the variables x and y as $x = r\cos\theta$ and $y = r\sin\theta$. Use the chain rule to calculate $\partial f/\partial r$ and $\partial f/\partial \theta$. ***(6 marks)***

End of Question Paper

Formula sheet

- **Trigonometric identities**

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

- **The local truncation error in the case of the 4th order Runge-Kutta method is given by**

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \le x \le l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\; dx, \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\; dx \quad (n = 1, 2, 3, \ldots)$$

If the function $f(x)$ is defined over the interval $0 \le x \le l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \ldots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \ldots)$$

- **the orthogonality of the sine function can be defined as**

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \textbf{if } m \ne n \\ L/2 & \textbf{if } m = n \end{cases}$$