The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2018-2019

Mathematics for Engineering Modelling

2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Derive the Maclaurin series for $\sinh(x)$ up to the x^5 term and for $\cosh(x)$ up to the x^4 term. *Use the series* to show that, up to and including terms in x^5,

$$2\sinh(x)\cosh(x) = \sinh(2x).$$

(10 marks)

(ii) Evaluate

$$\lim_{x\to\infty}(x^2 e^{-2x}).$$

(5 marks)

(iii) *Derive* the partial sum s_n to n terms of the infinite series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \qquad (r \neq 1).$$

Hence show that the series is convergent for $| r |< 1$ and find the sum.

(5 marks)

(iv) Find the sum of the infinite geometric series

$$2 + 6x + 18x^2 + 54x^3 + \cdots + 2(3x)^n + \cdots, \tag{1}$$

and state the value of the radius of convergence, R, of the series. Hence find, for $| x |< R$, the sum of the infinite series

$$6 + 36x + 162x^2 + \cdots + 6n(3x)^{n-1} + \cdots \tag{2}$$

(5 marks)

2 (i) Denote the Fourier series of

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ 0, & 0 \le x < \pi \end{cases}$$

by $F(x)$. Show that

$$F(x) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{\cos[(2m+1)x]}{(2m+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx).$$

(16 marks)

(ii) Sketch the graph of $F(x)$ for $-3\pi < x < 3\pi$.

(4 marks)

(iii) Use the expression for $F(x)$ to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

(5 marks)

3 (i) Find *by integration* the Laplace transform $F(s)$ of t^2. (You may assume that $s > 0$.)

(3 marks)

(ii) Find the constants A, B such that

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{A}{s^2+1} + \frac{B}{s^2+9}.$$

(4 marks)

(iii) Use the method of Laplace transforms to solve the ordinary differential equation

$$\ddot{y} + y = \cos(3t) + \sin(3t),$$

subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = 0$.

(9 marks)

(iv) In a coupled system, the variables $y_1(t)$, $y_2(t)$ satisfy the ordinary differential equations

$$\begin{aligned} \dot{y}_1 + y_2 &= 0 \\ \dot{y}_2 - 4y_1 &= 0, \end{aligned}$$

and the initial conditions $y_1(0) = 3$, $y_2(0) = 0$. Using Laplace transforms, solve for $y_1(t)$. Then determine $y_2(t)$ *without* calculating the Laplace transform of $y_2(t)$.

(9 marks)

4 (i) *Verify* that the d'Alembert general solution

$$u(x,t) = f(x-ct) + g(x+ct),$$

where f and g are arbitrary functions and c is a constant, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \qquad (3)$$

Hence find the transverse displacement, for time $t > 0$, of an infinite stretched string that, at $t = 0$, is at rest with displacement $\sin(x)$, where x is the distance along the string.

(10 marks)

(ii) The transverse displacement u of a stretched string, held fixed at its end-points $x = 0$ and $x = L$, satisfies the wave equation given by (3). At $t = 0$ the displacement is zero. *Verify* that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x)\sin(\alpha_n c t)$$

satisfies equation (3) and the initial and boundary conditions, where B_n and α_n $(n = 1, 2, 3, \ldots)$ are constants to be determined.

If it is further given that, at $t = 0$, the velocity $\partial u/\partial t = f(x)$, find an integral formula for B_n.

(15 marks)

5 (i) Evaluate

$$\int_{x=1}^{x=4}\int_{y=-1}^{y=2} (2x + 6x^2 y)\,dy\,dx.$$

(5 marks)

(ii) Use polar coordinates to evaluate

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} dy\,dx.$$

(8 marks)

(iii) Sketch the region of integration of the double integral

$$I = \int_{y=0}^{1}\int_{x=y}^{1} (1-x^2)^4\,dx\,dy.$$

By changing the order of integration evaluate I. ***(12 marks)***

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1 - r^n)/(1 - r), \quad r \neq 1 .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a = 0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!}f''(0)\,x^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where
$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx \ ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\, \cos\left(\frac{n\pi x}{l}\right) dx \ , \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\, \sin\left(\frac{n\pi x}{l}\right) dx \ , \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt \ .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} .$$

2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(xy)$ about $x = a, y = b$ is

$$f(xy) = f(a,b) + \{(x-a) f_x(a,b) + (y-b) f_y(a,b)\} +$$

$$+ \frac{1}{2!}\Big\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) +$$

$$+ (y-b)^2 f_{yy}(a,b)\Big\} +$$

$$+ \ldots$$

Here $f_x = \dfrac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn. equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A \cos \omega x + B \sin \omega x$$
$$X'' = \omega^2 X \Rightarrow X(x) = C \cosh \omega x + D \sinh \omega x$$
$$\text{or } Ee^{\omega x} + Fe^{-\omega x}$$
$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(iv)}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!(n \geq 1)$	$1/s^n$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{1}{a}\sin at$	$\dfrac{1}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\dfrac{1}{a}\sinh at$	$\dfrac{1}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
$\dfrac{\sin at - at\cos at}{2a^3}$	$\dfrac{1}{(s^2+a^2)^2}$
$\dfrac{t\sin at}{2a}$	$\dfrac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)f(t-a)$	$e^{-as}F(s)$, where $F(s) = L(f(t))$