The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2018–2019

Continuum Mechanics 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Write down in full the following expressions:

$$\text{(a)}\ \delta_{ij}U_{ij}, \quad \text{(b)}\ t_i = \varepsilon_{ijk}T_{jk}, \quad \text{(c)}\ V = \varepsilon_{ijk}u_i v_j w_k.$$

(6 marks)

(ii) Use the relation $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ to prove the identity

$$\nabla^2 \boldsymbol{a} = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla \times \nabla \times \boldsymbol{a}.$$

(9 marks)

(iii) New Cartesian coordinates, x_1', x_2', x_3', are obtained from the old ones, x_1, x_2, x_3, by a rotation about the x_3-axis through an angle θ. The transformation matrix from the old to the new coordinates is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix of tensor $\mathbf{T}$ in the old coordinates is given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Calculate the matrix of tensor $\mathbf{T}$ in the new coordinates.

[You can use without proof the relation $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$ between the matrices of tensor $\mathbf{T}$ in the old and new coordinates.] ***(7 marks)***

(b) You are given that the matrix of tensor $\mathbf{T}$ in the new coordinates is diagonal and the angle θ is positive and acute. Determine θ.

(3 marks)

2 (i) Derive the mass conservation equation in Lagrangian coordinates,

$$\rho(\boldsymbol{\xi}, t)\mathcal{J}(\boldsymbol{\xi}, t) = \rho_0(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ are the Lagrangian coordinates, ρ_0 and ρ are the density at the initial time, $t = 0$, and current time t respectively, and $\mathcal{J}$ is the Jacobian of transformation from the initial coordinates (ξ_1, ξ_2, ξ_3) to the current coordinates (x_1, x_2, x_3),

$$\mathcal{J}(\boldsymbol{\xi}, t) = \frac{D(x_1, x_2, x_3)}{D(\xi_1, \xi_2, \xi_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}.$$

(9 marks)

(ii) A body is subject to the deformation

$$x_1 = f(\xi_1)e^{-k\xi_3}, \quad x_2 = \xi_2, \quad x_3 = f(\xi_1)e^{k\xi_3},$$

where $f(\xi_1)$ is a positive, differentiable, monotonically increasing function, and k is a positive constant.

(a) Show that, after the deformation, the density ρ is given by

$$\rho = \frac{\rho_0}{2kf(\xi_1)f'(\xi_1)},$$

where ρ_0 is the density in the initial state. ***(6 marks)***

(b) You are given that $\xi_1 \geq 0$, and $x_1 = 1/k$ when $\xi_1 = \xi_3 = 0$. Determine the form of the function $f(\xi_1)$ if the density does not change during the deformation. ***(10 marks)***

3 (i) Write down the expression for the surface traction, $\boldsymbol{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\boldsymbol{n}$. Express it both in the vector and coordinate form. ***(2 marks)***

(ii) You are given that S is the surface of a simply connected volume V.

(a) Introducing the notation $a_l = x_j T_{kl}$ and using Gauss's theorem show that

$$\int_S \boldsymbol{x} \times \boldsymbol{t}\, dS = \boldsymbol{e}_i \varepsilon_{ijk} \int_V \left(T_{kj} + x_j \frac{\partial T_{kl}}{\partial x_l} \right) dV. \qquad (*)$$

(5 marks)

(b) Use equation $(*)$ and the momentum equation

$$\frac{d}{dt} \int_V \rho \boldsymbol{x} \times \boldsymbol{v}\, dV = \int_S \boldsymbol{x} \times \boldsymbol{t}\, dS + \int_V \rho \boldsymbol{x} \times \boldsymbol{b}\, dV$$

to prove that $\mathbf{T}$ is a symmetric tensor.

[You can use without proof the relation $\dfrac{d}{dt} \displaystyle\int_V f \rho\, dV = \int_V \rho \frac{\partial f}{\partial t}\, dV$, where V is a volume frozen in a continuum]

(9 marks)

(iii) You are given that in a continuum that is in equilibrium the stress has the form $T_{ij} = -p\delta_{ij}$, where δ_{ij} is the Kronecker delta symbol and p is the pressure.

(a) Show that in this case the equilibrium equation

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0$$

reduces to

$$\frac{\partial p}{\partial x_i} = \rho b_i.$$

(2 marks)

(b) You are also given that the continuum occupies the half-space $z \geq 0$ in Cartesian coordinates x, y, z. There is a constant body force in the negative z-direction, $\boldsymbol{b} = (0, 0, -g)$. The pressure p is related to the continuum density ρ by $p = p_0(\rho/\rho_0)^\alpha$, where $0 < \alpha < 1$ is a constant, and $p = p_0 =$ const and $\rho = \rho_0 =$ const at $z = 0$. Determine the dependence of p and ρ on z. ***(7 marks)***

4 (i) The motion of an ideal fluid is called potential if the velocity can be written in the form $\mathbf{v} = \nabla\varphi$, φ being called the velocity potential. By using Euler's equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \Pi\right),$$

where Π is the body force potential, $\boldsymbol{b} = -\nabla\Pi$, derive the Lagrange-Cauchy integral for fluid potential motion,

$$\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\|\mathbf{v}\|^2 + \Pi = f(t),$$

where $f(t)$ is an arbitrary function of time. ***(5 marks)***

(ii) You are given that water occupies a half-space $z < 0$ in Cartesian coordinates x, y, z with the z-axis anti-parallel to the gravity acceleration $\boldsymbol{g}$. The water is in equilibrium. Use the Lagrange-Cauchy integral to show that the water pressure is given by

$$p = p_a + g\rho h,$$

where p_a is the atmospheric pressure (i.e. $p = p_a$ at $z = 0$), ρ is the water density, and $h = -z$ is the water depth. ***(4 marks)***

(iii) Prove Archimedes' law: the pressure force exerted on the surface of a body immersed in water is in the vertical direction, and its magnitude is equal to the weight of water displaced by the body. ***(11 marks)***

(iv) A bathisphere has a spherical shape with the radius $R = 1$ m. Its mass is 5×10^3 kg. The bathisphere is attached to a floating ship by a steel rope and immersed in water. What is the tension in the rope?

[You can take the gravity acceleration $g = 10\ \mathrm{m\,s^{-2}}$ and the water density $\rho = 10^3\ \mathrm{kg\,m^{-3}}$.] ***(5 marks)***

5 (i) You are given that, in equilibrium, the stress tensor $\mathbf{T}$ satisfies the equation written in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \qquad (*)$$

where T_{ij} are the components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\boldsymbol{b}$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

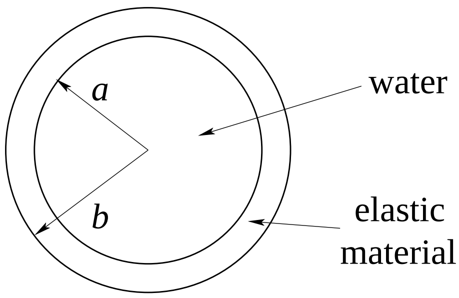
where u_i are the Cartesian components of the displacement $\boldsymbol{u}$, and λ and μ are the Lamé constants. Show that, in the linear elasticity, equation $(*)$ reduces to

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{b} = 0. \qquad (\dagger)$$

(4 marks)

(ii) There is an elastic spherical shell of internal radius a and external radius b (see the figure). The space inside the shell was filled with water through a small hole, and after that the hole was tightly sealed. Then the shell was put in a cold place with the temperature below zero. As a result the water froze and turned into ice. Calculate the internal radius of the shell, R, after the water froze.

[You can take the densities of water and ice equal to $\rho \approx 1000\,\mathrm{kg\,m^{-3}}$ and $\rho_i \approx 917\,\mathrm{kg\,m^{-3}}$ respectively.] ***(4 marks)***



(a) You are given that there is no body force, $\boldsymbol{b} = 0$. You can assume that inside the shell the displacement vector is $\boldsymbol{u} = u(r)\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit vector in the radial direction in the spherical coordinate system r, θ, ϕ with the origin at the shell centre. Use equation $(\dagger)$ to show that

$$\nabla(\nabla \cdot \boldsymbol{u}) = 0.$$

Show that the displacement in the shell is given by

$$u = Ar + \frac{B}{r^2},$$

where A and B are constants.

$\Big[$You can use without proof that, for $\boldsymbol{u} = u(r)\boldsymbol{e}_r$, $\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u})$ and $\nabla \cdot \boldsymbol{u} = \dfrac{1}{r^2}\dfrac{d(r^2 u)}{dr}$.$\Big]$ ***(5 marks)***

 Question 5 continued on next page

5 (continued)

(b) You are given that the surface traction at the external boundary of the shell is given by

$$\boldsymbol{t}(b) = \left(\frac{\lambda}{r^2} \frac{d(r^2 u)}{dr} \bigg|_{r=b} + 2\mu \frac{du}{dr} \bigg|_{r=b} \right) \boldsymbol{e}_r .$$

Use the boundary conditions $u = R - a$ at $r = a$ and $\boldsymbol{t} = 0$ at $r = b$ to calculate A and B, where R was calculated in part (ii). Determine the external radius of the deformed shell if $a = 10$ cm, $b = 12$ cm, and $\lambda = \mu$. ***(12 marks)***

End of Question Paper