Continuum Mechanics

Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

(i) Write down in full the following expressions:

(a) \( \delta_{ij}U_{ij} \),
(b) \( t_i = \varepsilon_{ijk}T_{jk} \),
(c) \( V = \varepsilon_{ijk}u_i v_j w_k \).

(6 marks)

(ii) Use the relation \( \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \) to prove the identity

\( \nabla^2 a = \nabla(\nabla \cdot a) - \nabla \times \nabla \times a. \)

(9 marks)

(iii) New Cartesian coordinates, \( x_1', x_2', x_3' \), are obtained from the old ones, \( x_1, x_2, x_3 \), by a rotation about the \( x_3 \)-axis through an angle \( \theta \). The transformation matrix from the old to the new coordinates is given by

\[
\hat{A} = \begin{pmatrix}
\cos\theta & \sin\theta & 0 \\
-\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The matrix of tensor \( T \) in the old coordinates is given by

\[
\hat{T} = \begin{pmatrix}
3 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(a) Calculate the matrix of tensor \( T \) in the new coordinates.

[You can use without proof the relation \( \hat{T}' = \hat{A}\hat{T}\hat{A}^T \) between the matrices of tensor \( T \) in the old and new coordinates.] (7 marks)

(b) You are given that the matrix of tensor \( T \) in the new coordinates is diagonal and the angle \( \theta \) is positive and acute. Determine \( \theta \). (3 marks)
(i) Derive the mass conservation equation in Lagrangian coordinates,

$$\rho(\xi, t)J(\xi, t) = \rho_0(\xi),$$

where $$\xi = (\xi_1, \xi_2, \xi_3)$$ are the Lagrangian coordinates, $$\rho_0$$ and $$\rho$$ are the density at the initial time, $$t = 0$$, and current time $$t$$ respectively, and $$J$$ is the Jacobian of transformation from the initial coordinates $$(\xi_1, \xi_2, \xi_3)$$ to the current coordinates $$(x_1, x_2, x_3)$$.

$$J(\xi, t) = \frac{D(x_1, x_2, x_3)}{D(\xi_1, \xi_2, \xi_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}.$$  

(9 marks)

(ii) A body is subject to the deformation

$$x_1 = f(\xi_1)e^{-k\xi_3}, \quad x_2 = \xi_2, \quad x_3 = f(\xi_1)e^{k\xi_3},$$

where $$f(\xi_1)$$ is a positive, differentiable, monotonically increasing function, and $$k$$ is a positive constant.

(a) Show that, after the deformation, the density $$\rho$$ is given by

$$\rho = \frac{\rho_0}{2kf(\xi_1)f'(\xi_1)},$$

where $$\rho_0$$ is the density in the initial state.  

(6 marks)

(b) You are given that $$\xi_1 \geq 0$$, and $$x_1 = 1/k$$ when $$\xi_1 = \xi_3 = 0$$. Determine the form of the function $$f(\xi_1)$$ if the density does not change during the deformation.  

(10 marks)
(i) Write down the expression for the surface traction, \( t \), in terms of the stress tensor, \( T \), and the unit normal to the surface, \( n \). Express it both in the vector and coordinate form.

(2 marks)

(ii) You are given that \( S \) is the surface of a simply connected volume \( V \).

(a) Introducing the notation \( a_l = x_j T_{kl} \) and using Gauss’s theorem show that

\[
\int_S x \times t \, dS = e_i \varepsilon_{ijk} \int_V \left( T_{kj} + x_j \frac{\partial T_{kl}}{\partial x_l} \right) \, dV.
\]

(5 marks)

(b) Use equation (\( * \)) and the momentum equation

\[
\frac{d}{dt} \int_V \rho x \times v \, dV = \int_S x \times t \, dS + \int_V \rho x \times b \, dV
\]

to prove that \( T \) is a symmetric tensor.

[You can use without proof the relation \( \frac{d}{dt} \int_V f \rho \, dV = \int_V \rho \frac{\partial f}{\partial t} \, dV \), where \( V \) is a volume frozen in a continuum]

(9 marks)

(iii) You are given that in a continuum that is in equilibrium the stress has the form \( T_{ij} = -p \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta symbol and \( p \) is the pressure.

(a) Show that in this case the equilibrium equation

\[
\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0
\]

reduces to

\[
\frac{\partial p}{\partial x_i} = \rho b_i.
\]

(2 marks)

(b) You are also given that the continuum occupies the half-space \( z \geq 0 \) in Cartesian coordinates \( x, y, z \). There is a constant body force in the negative \( z \)-direction, \( b = (0, 0, -g) \). The pressure \( p \) is related to the continuum density \( \rho \) by \( p = \rho_0(\rho/\rho_0)^\alpha \), where \( 0 < \alpha < 1 \) is a constant, and \( p = \rho_0 = \text{const} \) and \( \rho = \rho_0 = \text{const} \) at \( z = 0 \). Determine the dependence of \( p \) and \( \rho \) on \( z \).

(7 marks)
The motion of an ideal fluid is called potential if the velocity can be written in the form $\mathbf{v} = \nabla \varphi$, $\varphi$ being called the velocity potential. By using Euler’s equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \left( \frac{\rho}{\rho} + \frac{1}{2} \|\mathbf{v}\|^2 + \Pi \right),$$

where $\Pi$ is the body force potential, $b = -\nabla \Pi$, derive the Lagrange-Cauchy integral for fluid potential motion,

$$\frac{\partial \varphi}{\partial t} + \frac{\rho}{\rho} + \frac{1}{2} \|\mathbf{v}\|^2 + \Pi = f(t),$$

where $f(t)$ is an arbitrary function of time. \hspace{1cm} (5 marks)

You are given that water occupies a half-space $z < 0$ in Cartesian coordinates $x, y, z$ with the $z$-axis anti-parallel to the gravity acceleration $g$. The water is in equilibrium. Use the Lagrange-Cauchy integral to show that the water pressure is given by

$$p = p_a + g \rho h,$$

where $p_a$ is the atmospheric pressure (i.e. $p = p_a$ at $z = 0$), $\rho$ is the water density, and $h = -z$ is the water depth. \hspace{1cm} (4 marks)

Prove Archimedes’ law: the pressure force exerted on the surface of a body immersed in water is in the vertical direction, and its magnitude is equal to the weight of water displaced by the body. \hspace{1cm} (11 marks)

A bathisphere has a spherical shape with the radius $R = 1$ m. Its mass is $5 \times 10^3$ kg. The bathisphere is attached to a floating ship by a steel rope and immersed in water. What is the tension in the rope?

[You can take the gravity acceleration $g = 10$ m s$^{-2}$ and the water density $\rho = 10^3$ kg m$^{-3}$.] \hspace{1cm} (5 marks)
5 (i) You are given that, in equilibrium, the stress tensor $T$ satisfies the equation written in Cartesian coordinates $x_1, x_2, x_3$,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \quad (\ast)$$

where $T_{ij}$ are the components of the stress tensor $T$, $\rho$ is the density, and $b_i$ are the components of the body force $b$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where $u_i$ are the Cartesian components of the displacement $u$, and $\lambda$ and $\mu$ are the Lamé constants. Show that, in the linear elasticity, equation $(\ast)$ reduces to

$$(\lambda + \mu) \nabla (\nabla \cdot u) + \mu \nabla^2 u + \rho b = 0. \quad (\dagger)$$

(4 marks)

(ii) There is an elastic spherical shell of internal radius $a$ and external radius $b$ (see the figure). The space inside the shell was filled with water through a small hole, and after that the hole was tightly sealed. Then the shell was put in a cold place with the temperature below zero. As a result the water froze and turned into ice. Calculate the internal radius of the shell, $R$, after the water froze.

[You can take the densities of water and ice equal to $\rho \approx 1000 \text{ kg m}^{-3}$ and $\rho_i \approx 917 \text{ kg m}^{-3}$ respectively.] (4 marks)

![Diagram of a spherical shell with water inside and elastic material outside]

(a) You are given that there is no body force, $b = 0$. You can assume that inside the shell the displacement vector is $u = u(r)e_r$, where $e_r$ is the unit vector in the radial direction in the spherical coordinate system $r, \theta, \phi$ with the origin at the shell centre. Use equation $(\dagger)$ to show that

$$\nabla (\nabla \cdot u) = 0.$$

Show that the displacement in the shell is given by

$$u = Ar + \frac{B}{r^2},$$

where $A$ and $B$ are constants.

[You can use without proof that, for $u = u(r)e_r$, $\nabla^2 u = \nabla(\nabla \cdot u)$ and $\nabla \cdot u = \frac{1}{r^2} \frac{d}{dr} \left( r^2 u \right).$] (5 marks)
(b) You are given that the surface traction at the external boundary of the shell is given by

\[ t(b) = \left( \lambda \frac{d(r^2u)}{dr} \bigg|_{r=b} + 2\mu \frac{du}{dr} \bigg|_{r=b} \right) e_r. \]

Use the boundary conditions \( u = R - a \) at \( r = a \) and \( t = 0 \) at \( r = b \) to calculate \( A \) and \( B \), where \( R \) was calculated in part (ii). Determine the external radius of the deformed shell if \( a = 10 \) cm, \( b = 12 \) cm, and \( \lambda = \mu \).

(12 marks)

End of Question Paper