The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Introduction to Relativity **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Define an inertial frame of reference.
Give one example of an inertial frame and one example of a non-inertial frame. ***(3 marks)***

(ii) The observer A is stationary in an inertial frame $R : (ct, x)$ at $x = 0$.
The observer B is stationary in an inertial frame $\widetilde{R} : (c\widetilde{t}, \widetilde{x})$ at $\widetilde{x} = 0$.
$\widetilde{R}$ is moving relative to R at a velocity v in the x-direction, such that

$$\begin{pmatrix} c\widetilde{t} \\ \widetilde{x} \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma_v \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

The observers A and B pass each other at $t = \widetilde{t} = 0$.
Two lamps L_1 and L_2 are stationary in R at the positions $x = -\ell$ and $x = +\ell$, respectively.
The lamps are turned on at events E_1 and E_2 in such a way that the light from E_1 and E_2 reaches A and B at precisely $t = \widetilde{t} = 0$.

(a) Write down the coordinates (ct, x) of the events E_1 and E_2, and explain why they are simultaneous according to A. ***(3 marks)***

(b) Apply the Lorentz transformation to find the coordinates of the events $\widetilde{E}_1$ and $\widetilde{E}_2$ in $\widetilde{R}$.
(3 marks)

(c) Find the time interval between the events $\tilde{E}_1$ and $\tilde{E}_2$ according to B. Which event occurs first in $\tilde{R}$? ***(4 marks)***

(d) Write down the distance between the lamps according to the second observer B. ***(3 marks)***

(e) Draw a *spacetime diagram* showing the axes of R and $\widetilde{R}$; the world-lines of L_1 and L_2; the events E_1, E_2; and a pair of light rays passing through E_1 and E_2, respectively. ***(5 marks)***

2 Two inertial frames $R : (ct, x, y, z)$ and $\widetilde{R} : (c\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$ are related by the transformation

$$\begin{pmatrix} c\widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where L is a 4×4 matrix, and $g = \text{diag}[1, -1, -1, -1]$ is a 4×4 matrix representing the metric tensor.

(i) Define the conditions for L to represent a proper, orthochronous Lorentz transformation. ***(3 marks)***

(ii) Let the matrices L and M represent proper Lorentz transformations. Show that the matrix LM also represents a proper Lorentz transformation. ***(5 marks)***

(iii) Now let

$$L = \begin{pmatrix} \cosh\rho & 0 & 0 & \sinh\rho \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sinh\rho & 0 & 0 & \cosh\rho \end{pmatrix},$$

where ρ and θ are constants.
Show that L is a *non-proper*, orthochronous Lorentz transformation. ***(14 marks)***

(iv) Write down the velocity at which $\tilde{R}$ is moving with respect to R, for the transformation of part (iii).

(3 marks)

3 (i) (a) State the definition of a four-vector.
Define the Lorentz bracket $g(X, Y)$, where X and Y are four-vectors.
(4 marks)

(b) Define what is meant by a *future-pointing and time-like* four vector.
(3 marks)

(ii) Three events in an inertial frame R have space-time coordinates (ct, x, y, z) given by

$$A : (2, 1, 1, 1), \qquad B : (3, 0, 0, 1), \qquad C : (0, 0, 1, 1).$$

(a) Which of the following *displacement four-vectors* is time-like and future-pointing?
(1) from A to B, (2) from B to C, (3) from C to A.
(2 marks)

(b) There exists an inertial frame $\tilde{R}$ in which the time-like displacement four-vector identified in part (ii)(a) has components $(\alpha, 0, 0, 0)$.
Find the value of α. ***(3 marks)***

(iii) Let X be a future-pointing *time-like* four-vector, and let Y be a future-pointing *null* four-vector.

(a) Show that $g(X, Y) > 0$ in any inertial frame. ***(4 marks)***

(b) Hence show that $X + aY$ is future-pointing and time-like for any positive real constant a. ***(4 marks)***

(c) Show that $Y - bX$ is time-like for any positive value of b such that $b > 2g(X, Y)/\alpha^2$; and for any negative value of b. ***(5 marks)***

4 (i) A particle has worldline $X(t) = (ct, \mathbf{x}(t))$ in an inertial frame R.

(a) Define the quantities *proper time* τ, *four-velocity* V and *four-acceleration* A for this particle. ***(3 marks)***

(b) Show that the four-velocity V of the particle is

$$V = (\gamma_v c,\ \gamma_v \mathbf{v})$$

where $\mathbf{v} = \dfrac{d\mathbf{x}}{dt}$, $\gamma_v = \left(1 - v^2/c^2\right)^{-1/2}$ and $v^2 = \mathbf{v} \cdot \mathbf{v}$. ***(3 marks)***

(c) Show that $g(V, V) = c^2$. Classify V as timelike, spacelike or null. ***(3 marks)***

(d) Find $g(V, A)$. Hence show that

$$g(A, A) + g\left(V, \frac{dA}{d\tau}\right) = 0.$$

(5 marks)

(ii) A particle is in circular motion of radius r in the frame R with the displacement vector

$$X(\tau) = (c\tau \cosh\rho, 0, r\cos(\omega\tau), r\sin(\omega\tau))$$

where ρ and ω are positive constants.

(a) Find $V(\tau)$ and hence show that $\sinh\rho = \omega r/c$.
Show that the magnitude of the acceleration of the particle in its instantaneous rest frame is $a = r\omega^2$. ***(5 marks)***

(b) How many revolutions will the particle undergo in 1 second in R, if it is moving in a circular orbit of radius of $r = 3000\,\text{km}$ with $\omega = 100\,\text{s}^{-1}$?
(The speed of light is $c \approx 3 \times 10^8\,\text{ms}^{-1}$). ***(6 marks)***

5 (i) (a) Define the *rest mass* m and *four-momentum* P of a particle.

(2 marks)

(b) A particle of rest mass m is moving at velocity $\mathbf{v}$ in an inertial frame R. Write down expressions for the particle's energy E and three-momentum $\mathbf{p}$.
Hence show that

$$E^2 - p^2c^2 = m^2c^4.$$

where $p^2 = \mathbf{p} \cdot \mathbf{p}$. ***(5 marks)***

(ii) A particle of rest mass M_1 is moving at velocity $\mathbf{u}$ with Lorentz factor $\gamma_u = (1 - u^2/c^2)^{-1/2}$ in an inertial frame R. The particle then splits into two identical particles of rest mass m_2, such that the first new particle is at rest in R, and the second new particle moves at velocity $\mathbf{v}$ in R with Lorentz factor $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) Show that

$$\begin{aligned} \gamma_u M_1 &= m_2(1 + \gamma_v) \\ \gamma_u M_1 \, \mathbf{u} &= \gamma_v \, m_2 \, \mathbf{v}. \end{aligned}$$

(5 marks)

(b) Show that $M_1 = 2\,\gamma_u\, m_2$.

(5 marks)

(c) Show that $\mathbf{v} = f(u)\mathbf{u}$ where $f(u)$ is some function you should determine. Hence show that the second new particle must be travelling faster than the old particle in R. ***(8 marks)***

End of Question Paper