The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS | Autumn Semester 2018–19

WAVES

2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

Please leave this exam paper on your desk
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to be completed by student

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1 (i) *Derive* d'Alembert's general solution for the one-dimensional wave equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}$$

on $-\infty < x < \infty$ for $t \geq 0$.

(13 marks)

(ii) Find $\phi(x, t)$, given that $c = 1$ and at $t = 0$

$$\phi(x, 0) = \sin kx, \qquad \frac{\partial\phi}{\partial t} = -k\cos kx,$$

where k is a constant.

(7 marks)

(iii) Give a physical interpretation of your solution. Further, explain why the solution, subject to the initial conditions in (ii), cannot (or can) be a standing wave.

(5 marks)

2 The position of a vibrating string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The ends of the string at $x = 0$ and $x = a$ are fixed at $u = 0$, and initially the string is at rest.

(i) Using the *method of separation of variables*, show that this configuration has the general solution

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{a}x \cos\frac{n\pi}{a}ct.$$

(16 marks)

(ii) If $a = \pi$ and initially $u = \varepsilon\sin(3x)\cos(x)$, where ε is a constant, determine the A_n for this case.

(5 marks)

(iii) Show that the general solution above in (i) can be expressed in terms of waves travelling to the left and right at speed c.

(4 marks)

3 In a compressible and uniform fluid the equilibrium density and pressure are ρ_0 and p_0, respectively. Due to the passage of a compressible perturbation, there are *linear* changes in density ρ and pressure p, and the resulting fluid velocity is $\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}$, where u, v and w are small.

(i) Given that the exact equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

obtain a *linear* approximation to this equation.

(4 marks)

(ii) In the *linear* approximation Newton's Second Law can be given by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

plus two similar equations. Given that $p \equiv p(\rho)$ only, show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right),$$

where c^2 is a constant which should be defined. Here, you may use the result from (i) in the derivation.

(8 marks)

(iii) In a particular case $\rho = \rho(r, t)$, where (as usual) $r^2 = x^2 + y^2 + z^2$. By changing the variables from Cartesians to spherical polars show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial r^2} + \frac{2}{r}\frac{\partial \rho}{\partial r}\right). \qquad (*)$$

Find the partial differential equation satisfied by $R = r\rho$, and hence write down the general solution of $(*)$. (You may quote d'Alembert's general solution of the one-dimensional wave equation.)

(13 marks)

4 The equilibrium position of the free surface of a liquid of depth h is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the free surface and

$$\eta = \eta_0 sin(kx - \omega t),$$

where η_0, k and ω are positive constants. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\phi_{xx} + \phi_{zz} = 0.$$

You are also given that (a) $\phi_z \to 0$ as $z \to -h$; (b) $\phi_z = \eta_t$ at $z = 0$; (c) $\phi_t + g\eta = 0$ at $z = 0$, where g is the constant acceleration of gravity.

(i) Give a brief physical interpretation of (a), (b) and (c).

(3 marks)

(ii) Find $\phi(x, z, t)$ and show that

$$\omega^2 = gk \tanh(kh).$$

(11 marks)

(iii) Consider a particle whose equilibrium position is (x_0, z_0). Suppose its position at time t is $(x_0 + X(t), z_0 + Z(t))$, where the time means of X and Z will be chosen to be zero. Using the velocity potential $\phi(x, z, t)$ and dispersion relation derived in (ii) above, show that

$$\frac{X^2}{a^2} + \frac{Z^2}{b^2} = 1$$

and determine a and b. Explain briefly the result.

(11 marks)

5 (i) Using the method of characteristics integrate the *associated equations* to derive the solution of

$$x^3 z_x = z_y$$

with $z = 1/(1+x^2)$ on $y = 0$, $-\infty < x < \infty$. Explain why the solution is not defined when $y \geq 1/(2x^2)$.

(8 marks)

(ii) Using the method of characteristics solve

$$\rho_t + \rho\rho_x = 0$$

with $\rho = f(x)$ at $t = 0$ given in two cases:

(a) $f(x) = 0 \quad (x < 0), \quad f(x) = x \quad (0 \leq x < 1), \quad f(x) = 1 \quad (x \geq 1);$
(b) $f(x) = 0 \quad (x < 0), \quad f(x) = -x \quad (0 \leq x < 1), \quad f(x) = -1 \quad (x \geq 1).$

(9 marks)

(iii) Determine, whether the solutions found in (ii) break down for any time $t \geq 0$, and if so, at what value.

(2 marks)

(iv) Sketch the function $\rho(x)$ at $t = 0$ and the characteristics for cases (a) and (b), respectively, on $-3 \leq x \leq 3$.

(6 marks)

End of Question Paper