The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

MAS331 Metric Spaces **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Write down the axioms for a metric space (X, d). ***(3 marks)***

(ii) Let (X, d) be a metric space where the set X has at least two distinct elements, and write $\mathbb{R} = A \cup A^c$ where

$$A := \{\alpha \in \mathbb{R}; \alpha d \text{ is a metric on } X\},$$

and for each $\alpha \in \mathbb{R}, x, y \in X, (\alpha d)(x, y) := \alpha d(x, y)$.

Give a precise description of the set A. What goes wrong on A^c?

(3 marks)

(iii) The *taxicab metric* d_1 on $\mathbb{R}^k$ is given by

$$d_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|,$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Prove that d_1 is a metric.

(7 marks)

(iv) The *maximum metric* d_∞ on $\mathbb{R}^k$ is given by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_d - y_d|\},$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Prove that

$$d_\infty(x, y) \leq d_1(x, y) \leq k d_\infty(x, y).$$

(3 marks)

(v) Use the result of (iv) to show that a sequence (x_n) converges to x in $(\mathbb{R}^k, d_1)$ if and only if it converges to x in $(\mathbb{R}^k, d_\infty)$. ***(6 marks)***

(vi) What is the limit of the sequence $((1 - 1/n, \cos(\pi/n), 1 + 1/n, e^{-n^2}))$ as $n \to \infty$ in the metric space $(\mathbb{R}^4, d_\infty)$? Briefly justify your answer.

(3 marks)

2 (i) Explain what is meant in a metric space (X, d) by

(a) a closed ball $B[a, r]$, centred at a and having radius r;

(b) a closed set. ***(3 marks)***

(ii) Prove that every closed ball in (X, d) is a closed set in (X, d). ***(5 marks)***

(iii) Consider the metric space $(C[0, 1], d_1)$ where $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.
Which of the following functions is in the ball $B[0, 1]$?

(a) $f(x) = 2 - x$

(b) $g(x) = e^{-x}$.

Present working to justify your conclusions.

(6 marks)

(iv) (a) Write down the precise definition of what it means for a sequence (x_n) in a metric space (X, d) to converge to a limit x as $n \to \infty$. ***(2 marks)***

(b) Prove that the sequence (x_n) of (a) converges to x if and only if $\lim_{n\to\infty} d(x, x_n) = 0$ in $\mathbb{R}$. ***(2 marks)***

(v) Let $f \in C[0, 1]$ be arbitrary and define the sequence of functions (f_n) by $f_n(x) = f(x) + x^n$ for $x \in [0, 1]$. Show that (f_n) converges in $(C[0, 1], d_1)$ and write down the corresponding limit. What can you say about the limiting behaviour of the sequence (g_n) in the same metric space, where $g_n(x) = f(x)x^n$, for each $x \in [0, 1]$? ***(7 marks)***

3 (i) If (X_1, d_1) and (X_2, d_2) are metric spaces, explain using (a) sequences, (b) ϵ and δ, what it means for $f : X_1 \to X_2$ to be continuous. ***(4 marks)***

(ii) (a) Equip $\mathbb{R}^3$ with the usual (Euclidean) metric and $\mathbb{R}$ with its standard metric. Show that the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f((x, y, z)) = x^2 y^3 z,$$

is continuous. ***(3 marks)***

(b) Stating any results that you need from the course, use the result of (a) to show that

$$A := \{(x, y, z) \in \mathbb{R}^3;\ 2x^2 y^3 z > 1\}$$

is open, and that

$$B := \{(x, y, z) \in \mathbb{R}^3;\ 3x^2 y^3 z - 1 \leq 1/2\}$$

is closed. ***(6 marks)***

(iii) Let $g : [0, 1] \to \mathbb{R}$ be continuous and consider the mapping $M_g : C[0, 1] \to C[0, 1]$ given by

$$M_g f(x) = g(x) f(x),$$

for all $x \in [0, 1]$. Prove that M_g is continuous from $(C[0, 1], d_1)$ to $(C[0, 1], d_1)$ and from $(C[0, 1], d_\infty)$ to $(C[0, 1], d_1)$, where

$$d_1(f, h) = \int_0^1 |f(x) - h(x)| dx \text{ and } d_\infty(f, h) = \sup_{x \in [0,1]} |f(x) - h(x)|.$$

(6 marks)

(iv) (a) Explain what it means for a metric space (X, d) to be *compact.* ***(1 mark)***

(b) Let (X, d) be compact and $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ be a sequence of non–empty closed subsets of X. Deduce that

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset.$$

[Hint: Start with a sequence (x_n) such that $x_n \in F_n$ for all $n \in \mathbb{N}$.] ***(5 marks)***

4 (i) Explain what is meant by a *contraction* in a metric space. State the *contraction mapping theorem* ***(4 marks)***

(ii) If (X, d) is a metric space and f is a contraction on X, show that its nth iterate f^n is a contraction for all $n \in \mathbb{N}$, where $f^n = f \circ f \circ \cdots \circ f$ (n times). ***(4 marks)***

(iii) Show that if (X, d) is a complete metric space such that f^n is a contraction for *some* $n \in \mathbb{N}$, then f has a unique fixed point. ***(4 marks)***

(iv) Let $T : C[0,1] \to C[0,1]$ be given for each $f \in C[0,1], x \in [0,1]$ by

$$Tf(x) = 7 + \frac{1}{5}\int_0^x f(t)dt.$$

Prove that T is a contraction in $(C[0,1], d_\infty)$, where

$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ for all $f, g \in C[0,1]$, and hence deduce that T has a unique fixed point. ***(5 marks)***

(v) Using any results that you need from the course, find all values of $a \in \mathbb{R}$ such that $f(x) = \cos(ax)$ is a contraction on $[-\pi/2, \pi/2]$. Is it true that the equation $\cos(x) = x$ has a unique solution for $x \in [-\pi/2, \pi/2]$? Give reasons for your answer (a graph is not a suitable response).

[Hint: Use the result of (iii). You may also use the facts that $|\sin(x)| \leq 1$ and $|\sin(\cos(x))| < 1$ for all $x \in [-\pi/2, \pi/2]$.] ***(8 marks)***

5 (i) Explain what it means for a subset A of a metric space (X, d) to be *complete*. ***(2 marks)***

(ii) Prove that if X is complete, and $A \subset X$ is closed, then A is complete. ***(4 marks)***

(iii) Equip $\mathbb{R}$ with its standard metric: $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Which of the following subsets of $(\mathbb{R}, d)$ are complete:

(a) $[0, 1]$;

(b) $\mathbb{Q} \cap [0, 1]$;

(c) $\mathbb{Q} \cup [0, 1]$?

In each case give a justification for your answer. ***(6 marks)***

(iv) If A and B are complete subsets of a metric space, show that $A \cap B$ is complete. ***(3 marks)***

(v) The aim of this part is to prove that if A and B are complete subsets of a metric space, then $A \cup B$ is complete. Begin by taking (x_n) to be a Cauchy sequence in $A \cup B$. Then there are three possibilities:

(I) (x_n) has an infinite number of terms in A, and only finitely many in B;

(II) (x_n) has an infinite number of terms in B, and only finitely many in A.

(III) (x_n) has an infinite number of terms in each of A and B.

(a) In cases (I) and (II), deduce that (x_n) converges to a limit in $A \cup B$. ***(4 marks)***

(b) In case (III), show that (x_n) has a subsequence that converges to a limit $a \in A$, and a subsequence that converges to a limit $b \in B$. By using a limiting argument, deduce that $d(a, b) = 0$. What can you then conclude about a and b? ***(5 marks)***

[Hint: You may use the fact that subsequences of Cauchy sequences are themselves Cauchy.]

(c) Show that $A \cup B$ is complete. ***(1 mark)***

End of Question Paper