The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018-2019**

Complex Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) Let $z, w \in \mathbb{C}$. State, without proof, the triangle inequalities for $|\, z + w|$ and $|\, z - w|$. ***(2 marks)***

(b) Show that for all $z \in \mathbb{C}$ such that $|z| < 1$,

$$\left|\frac{z^3+3}{z^2+2z+5}\right| \geq \frac{1}{4}$$ ***(5 marks)***

(ii) Find all the solutions in $\mathbb{C}$ of the equation

$$\cosh z = -\frac{1}{2}.$$ ***(4 marks)***

(iii) Find all seven roots of the equation $z^7 + 1 = 0$ and hence express $z^7 + 1$ as a product of one real linear factor and three real quadratic factors. ***(6 marks)***

(iv) Let H be the half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let γ be any path in H with initial point 1 and final point $\sqrt{3}$. Prove that

$$\int_\gamma \frac{dz}{1+z^2} = \frac{\pi}{12} .$$ ***(8 marks)***

2 (i) Explain what is meant by the following statement: The function f is analytic in the region D. ***(3 marks)***

(ii) (a) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(2 marks)***

(b) Let a, b, and c be positive real numbers. Decide whether there exists a function $g = u(x,y) + iv(x,y)$ analytic in $\mathbb{C}$ such that $au(x,y) + bv(x,y) = c$ for all $x, y \in \mathbb{R}$. If g exists, determine what type of function it is. ***(7 marks)***

(iii) In each of the following cases, decide whether there is a function g analytic on $\mathbb{C}$ such that $\mathrm{Re}[g(x+iy)] = u(x,y)$. When g exists find an explicit expression for $g(z)$ in terms of z

(a) $u(x,y) = x^4 - 6x^2y^2 + y^4$, ***(2 marks)***

(b) $u(x,y) = 2\cos x \cosh y + 2x^2$, ***(2 marks)***

(c) $u(x,y) = 3x - 5\cos 2x \sinh 2y$. ***(9 marks)***

3 (i) State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

(ii) Let γ be the triangular contour with vertices $0,\ 4+4i,\ -4+4i$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(a) $$\int_\gamma \frac{z^4 \sin z^3}{(z+3)^2}\, dz\,,$$

(b) $$\int_\gamma \frac{e^z+1}{z^2+9}\, dz\,,$$

(c) $$\int_\gamma \frac{\cosh 2z}{(z-2i)^5}\, dz\,,$$

(d) $$\int_\gamma [\,\mathrm{Re}(z) \ + \ \mathrm{Im}(z)\,]\, dz\,.$$

(18 marks)

4 (i) State without proof Liouville's Theorem. ***(2 marks)***

The function f is analytic in $\mathbb{C}$ and satisfies the relation $|\,3+4f(z)\,| < 5\,|f(z)|$ for all $z \in \mathbb{C}$. Show that f is constant. ***(6 marks)***

(ii) Find all the singularities in $\mathbb{C}$ of each of the following functions. Classify them giving reasons for your answers. Find the residue at each of the singularities.

(a) $h(z) = \dfrac{\cos z}{z(1+e^z)}$; ***(6 marks)***

(b) $k(z) = z\cosh\left(\dfrac{1}{z-2}\right)$; ***(5 marks)***

(c) $m(z) = \dfrac{\cos^2(\pi z)}{(2z-1)^5}$. ***(6 marks)***

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

(ii) Let γ be the square contour with **vertices** $-i,\ 1,\ i,\ -1$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{1}{\cos(\pi z)}\,dz \qquad \int_\gamma (z^4+1)\sin\left(\frac{1}{z}\right)\,dz.$$

(8 marks)

(iii) Let $\alpha > 0$. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos\alpha x + \sin\alpha x}{x^2+2x+5}\,dx$$

(13 marks)

End of Question Paper