The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018-19**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let L be a field and K be a subset of L. What is meant by saying that K is a subfield of L? ***(3 marks)***

(ii) For each of the subsets J_1 and J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{7} : \ a, b \in \mathbb{Q}\}$. ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + i\sqrt{3} : \ a, b, c \in \mathbb{Q}\}$ where $i = \sqrt{-1}$. ***(5 marks)***

(iii) (iii) Let $L = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ and $a = p\sqrt{3} + q\sqrt{5}$ where p and q are positive rational numbers.

(a) Show that $L = \mathbb{Q}(a)$. ***(5 marks)***

(b) Let $b = p\sqrt{3} - q\sqrt{5}$. Show that $b \neq 0$. ***(3 marks)***

(c) Express the element b^{-1} as a sum $\lambda_0 + \lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3$ where $\lambda_i \in \mathbb{Q}$. ***(4 marks)***

2 (i) Give a definition of n-th *cyclotomic polynomial* $\phi_n(x)$ where n is a natural number. ***(2 marks)***

(ii) Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$ ***(4 marks)***

provided p is a prime number.

(iii) Give an explicit expression for the polynomial $\phi_n(x)$ for $n = 1, 2, 3, 4$. ***(4 marks)***

(iv) Let p be a prime number. Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbb{Q}[x]$. ***(8 marks)***

(v) Let $K \subseteq L$ be a field extension, and let an element $a \in L$ be algebraic over the field K.

(a) Give the definition of the *minimal polynomial* of a over K. ***(2 marks)***

(b) Show that the minimal polynomial of a over K is an irreducible polynomial in $K[x]$. ***(2 marks)***

(c) Let f be a monic irreducible polynomial with $f(a) = 0$. Show that f is the minimal polynomial of a. ***(3 marks)***

3 (i) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(ii) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(2 marks)***

(iii) Prove the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(9 marks)***

(iv) Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$ be finite field extensions. Prove that

$$[K_n : K_0] = [K_n : K_{n-1}][K_{n-1} : K_{n-2}] \cdots [K_1 : K_0].$$ ***(5 marks)***

(v) Using (iv) (or otherwise) find $[\mathbb{Q}(\sqrt{3}, \sqrt{5}, i) : \mathbb{Q}]$. ***(7 marks)***

4 (i) Specify the four standard constructions that are used in the theory of ruler-and-compass constructions and involve perpendicular or parallel lines. ***(6 marks)***

(ii) Give the definition of a Fermat prime number. ***(2 marks)***

(iii) Explain in terms of constructible numbers what it means to say that a regular n-gon can be constructed. ***(3 marks)***

(iv) Prove that if p is an odd prime number which is not a Fermat prime then the regular p-gon cannot be constructed. (You may use the facts that the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$ is irreducible, a prime number of the form $2^k + 1$ is a Fermat prime). ***(8 marks)***

(v) Prove that if $m, n \geq 3$ are positive coprime integers such that the regular m-gon and the regular n-gon can be constructed then the regular mn-gon can be constructed. ***(6 marks)***

End of Question Paper