



The
University
Of
Sheffield.

MAS334

SCHOOL OF MATHEMATICS AND STATISTICS

**Autumn Semester
2018–19**

Combinatorics

2 hours 30 minutes

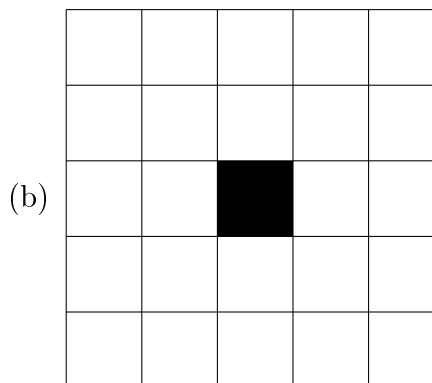
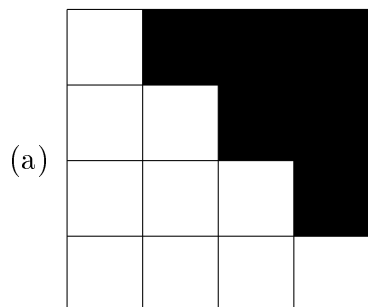
Attempt all the questions. The allocation of marks is shown in brackets.

- 1 (a) State the definition of the binomial coefficient $\binom{n}{k}$ in terms of factorials. *(1 mark)*
- (b) State Pascal's relation for binomial coefficients. *(2 marks)*
- (c) Suppose that $n \geq k \geq 2$. By thinking about the maximum and minimum elements of subsets of $\{1, \dots, n\}$, prove that *(8 marks)*

$$\binom{n}{k} = \sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^n \binom{j-i-1}{k-2}.$$

- 2 (a) Suppose that $m, k \geq 0$. How many solutions are there for $u_1 + \dots + u_k = m$ with u_1, \dots, u_k being nonnegative integers? *(2 marks)*
- (b) How many solutions are there for the equation $\sum_{i=1}^9 x_i = 48$, where each x_i is a natural number with $x_i \geq i$? *(7 marks)*

3 Consider the following two boards:



(The black squares do not count as part of the board.)

(a) Can board (a) be covered by non-overlapping dominos? Justify your answer. *(3 marks)*

(b) Can board (b) be covered by non-overlapping dominos? Justify your answer. *(3 marks)*

4 (a) State the inclusion-exclusion principle. *(3 marks)*

(b) Let B be the set of permutations σ of $\{1, \dots, 6\}$ such that either $\{\sigma(1), \sigma(2)\} = \{1, 2\}$, or $\{\sigma(3), \sigma(4)\} = \{3, 4\}$, or $\{\sigma(5), \sigma(6)\} = \{5, 6\}$. Use the inclusion-exclusion principle to find $|B|$. *(10 marks)*

5 Let A be a subset of \mathbb{Z} with $|A| = 10$. Show that there are disjoint subsets $B, C \subseteq A$ with $\sum B = \sum C \pmod{1000}$. *(9 marks)*

- 6 How many ways are there of placing six non-challenging rooks on the following board? (7 marks)

	1	2	3	4	5	6
<i>a</i>						
<i>b</i>						
<i>c</i>						
<i>d</i>						
<i>e</i>						
<i>f</i>						

Hint: you do not need to use any theorems; it is easier to argue directly.

- 7 Calculate the rook polynomial for the following board: (13 marks)

- 8 (a) State Landau's theorem on scores in tournaments. (4 marks)
- (b) Now consider a tournament of 8 players.
- (1) Can there be three players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.
 - (2) Can there be four players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.

(8 marks)

- 9 Find numbers a, \dots, x such that the following matrix becomes a latin square: *(8 marks)*

$$\left[\begin{array}{cccc|cc} \mathbf{a} & 3 & 2 & 4 & \mathbf{b} & \mathbf{c} \\ 2 & 4 & 6 & 5 & \mathbf{d} & \mathbf{e} \\ 4 & 6 & \mathbf{f} & 3 & \mathbf{g} & \mathbf{h} \\ 6 & 5 & 4 & 2 & \mathbf{i} & \mathbf{j} \\ \hline \mathbf{k} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{s} & \mathbf{t} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array} \right]$$

- 10 (a) Define what it means for two latin squares to be orthogonal. *(3 marks)*

- (b) It is given that the following matrices are latin squares, and that they are all orthogonal to each other:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix}$$

- (1) Using the orthogonality of A and C , show that none of b, g and m can be equal to 1. Find similar restrictions for the rest of the variables. *(3 marks)*
- (2) Use the orthogonality of B and C in the same way to find nine more restrictions. *(2 marks)*
- (3) Use the fact that C is a latin square to give nine more restrictions. *(2 marks)*
- (4) Find all the entries in C . *(2 marks)*

End of Question Paper