Data provided: list of formulae

The University Of Sheffield.

MAS336

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Differential Geometry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

A list of formulae is provided on the last two pages.

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) (a) If $\gamma\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$ is a unit-speed parametrized curve, state what it means for a function $\theta\colon\ ]\alpha,\beta[\ \to \mathbb{R}$ to be a *turning angle* for γ. Show that if $\theta\colon\ ]\alpha,\beta[\ \to \mathbb{R}$ is a turning angle for γ then the curvature κ of γ is given by

$$\kappa(t) = \theta'(t).$$

(5 marks)

(b) Consider the smooth curve $\gamma_0\colon\ ]0,\infty[\ \to \mathbb{R}^2$ given by

$$\gamma_0(t) = \tfrac{1}{2}(t\sin(\ln t) + t\cos(\ln t), t\sin(\ln t) - t\cos(\ln t)).$$

Verify that γ_0 is unit-speed and use the above part to show that the curvature κ of γ_0 is given by $\kappa(t) = 1/t$. ***(5 marks)***

(ii) (a) Given a smooth function $f\colon \mathbb{R} \to \mathbb{R}$, show how to construct a smooth parametrized curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ such that the curvature is equal to f. Explain why the curve you construct has this property. [Hint: You are only asked for a single curve, so you may fix constants as you see fit.] ***(5 marks)***

(b) Define the function $f_1\colon \mathbb{R} \to \mathbb{R}$ by $f_1(t) := \dfrac{1}{1+t^2}$. Using only the properties of the function f_1 and without doing any further calculation, sketch a curve whose curvature function is equal to the function f_1; indicate on your sketch the relevant features. ***(5 marks)***

(c) Use the construction you gave above to find the formula of a parametrized curve $\gamma_1\colon \mathbb{R} \to \mathbb{R}^2$ whose curvature function is f_1. You should simplify your answer as much as you can. [Hint: you might find the formula sheet useful.] ***(5 marks)***

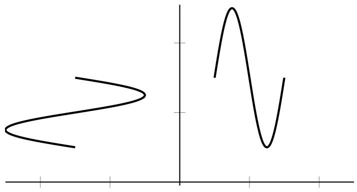
2 State whether each of the following assertions is true or false and give careful justification of your answer.

(i) The parametrized curve $\gamma_1 \colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma_1(t) = (\sin t, \sin t)$ parametrizes the level curve $\{(x,y) \in \mathbb{R}^2 \mid x - y = 0\}$. ***(4 marks)***

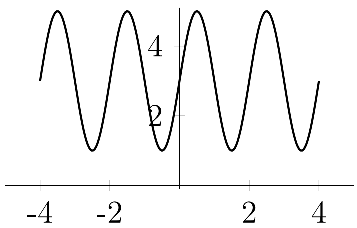
(ii) The parametrized curve $\gamma_2 \colon \;]-1,1[\; \to \mathbb{R}^2$ given by $t \mapsto (t, t^2+1)$ is a reparametrization of the curve $\gamma \colon \;]-1,1[\; \to \mathbb{R}^2$ given by $t \mapsto (t,t^2)$. ***(4 marks)***

(iii) The parametrized curve $\gamma_3 \colon \;]0,2[\; \to \mathbb{R}^2$ given by $t \mapsto (t-1,(t-1)^2)$ is a reparametrization of the curve $\gamma \colon \;]-1,1[\; \to \mathbb{R}^2$ given by $t \mapsto (t,t^2)$. ***(4 marks)***

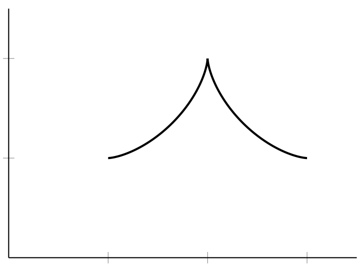
(iv) The two pictured curves are unit-speed parametrized curves $]\alpha,\beta[\; \to \mathbb{R}^2$ (for some $\alpha < \beta$) with the same curvature function. ***(4 marks)***



(v) The curve pictured below is the image of a *unit-speed* parametrized curve $\gamma_5 \colon \;]0,1[\; \to \mathbb{R}^2$. ***(4 marks)***



(vi) The following pictured curve is the image of a unit-speed parametrized curve. ***(4 marks)***



3 (i) Consider the curve $\gamma_0\colon \mathbb{R} \to \mathbb{R}^3$ given by $\gamma_0(t) = (\operatorname{sech}(t), 0, \tanh(t))$. Recalling that $\cosh^2(t) - \sinh^2(t) = 1$ for all t, give a relation satisfied by $\operatorname{sech}(t)$ and $\tanh(t)$ for all t. Hence state which well-known curve the image of γ_0 lies on. Without proving anything, state precisely or give a clear sketch of what the image of γ_0 is.
(4 marks)

(ii) Let $\gamma\colon\ ]\alpha, \beta[\to \mathbb{R}^3$, $t \mapsto (x(t), 0, z(t))$ be a smooth and regular parametrised curve. The parametrized surface of revolution, $\sigma\colon\ ]\alpha, \beta[\times \mathbb{R} \to \mathbb{R}^3$, obtained by rotating the image of γ around the z-axis, is given by

$$\sigma(t, \theta) := (\cos(\theta)x(t), \sin(\theta)x(t), z(t)).$$

(a) Define what it means for a parametrized surface to be regular.
(2 marks)

(b) Calculate $\sigma_t \times \sigma_\theta$. Hence, or otherwise, show that σ is regular if and only if $x(t) > 0$ for all t or $x(t) < 0$ for all t. ***(7 marks)***

(iii) Let $\sigma_0\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ be the surface of revolution of the curve γ_0 from above.

(a) Explain how you can deduce that σ_0 is regular. ***(2 marks)***

(b) Calculate the first fundamental form of σ_0. ***(3 marks)***

(c) Give the definition of a parametrized surface being conformal and explain how you can deduce that σ_0 is conformal. ***(4 marks)***

(d) State precisely (without proof) what the image of σ_0 is. Explain a real-world use of the fact that σ_0 is conformal. ***(4 marks)***

4 Consider the helicoid, given by the parametrized surface

$$\rho\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3; \quad (u, v) \mapsto (v\cos(u), v\sin(u), u).$$

(i) Show that the first and second fundamental forms of ρ at (u, v) are respectively

$$\begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & (1+v^2)^{-1/2} \\ (1+v^2)^{-1/2} & 0 \end{pmatrix}.$$

(7 marks)

(ii) Find the Weingarten matrix of ρ at (u, v) and show that the principal curvatures are $\pm(1+v^2)^{-1}$. ***(6 marks)***

(iii) For the principal curvature $(1+v^2)^{-1}$ find a corresponding principal direction and hence a corresponding principal vector. ***(7 marks)***

(iv) Prove that every point $\rho(u_0, v_0)$ on the helicoid lies on a straight line that is contained in the helicoid. Show that this can be used to deduce the fact, seen above, that the two principal curvatures at each point do not have the same sign. ***(5 marks)***

End of Question Paper

List of formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad-bc\neq 0$ is
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
- The cross-product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is
$$v_1 \times v_2 = (y_1z_2 - z_1y_2, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$
- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by
$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\,\|v_2\|}.$$

Inverse hyperbolic functions are given by the following.

- $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$
- $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$
- $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

Derivatives of some functions are given by the following.

- $\frac{d}{dx}\operatorname{sech}(x) = -\tanh(x)\operatorname{sech}(x)$
- $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$
- $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$
- $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$ for $|x| < 1$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

Some relations between trigonometric functions are the following.

- $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$
- $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

For a parametrized curve $\gamma\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ we have the following.

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is
$$\int_a^b \|\dot\gamma(t)\|\,dt.$$
- The curvature of γ at t is
$$\kappa(t) = \frac{\ddot\gamma(t)\cdot J(\dot\gamma(t))}{\|\dot\gamma(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$
where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the anti-clockwise rotation of angle $\pi/2$.

For a parametrized surface $\sigma\colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$ the following hold.

- The first fundamental form is given by
$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$
for all $(u,v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.
- The area of the domain $\sigma\left([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\right)$, for $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subseteq U$ is given by
$$\int_{u=\alpha_1}^{\beta_1} \int_{v=\alpha_2}^{\beta_2} \sqrt{EG - F^2}\, dv\, du$$
- The preferred unit normal vector along σ is given by $\mathbf{n}\colon U \to \mathbb{R}^3$,
$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$
- The second fundamental form of σ at $(u,v) \in U$ is
$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$
where $L = \sigma_{uu} \cdot \mathbf{n}$, $M = \sigma_{uv} \cdot \mathbf{n}$ and $N = \sigma_{vv} \cdot \mathbf{n}$.
- The Weingarten matrix of σ is
$$W = \mathrm{I}^{-1}\, \mathrm{II}\,.$$
- The Gaussian curvature is
$$K = \det W.$$

The Brioschi formula:
$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$