The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2018–19

Game Theory

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Alice and Bob participate in a first-price sealed-bid auction, by bidding £a and £b respectively, where a and b are integers satisfying $0 \leq a \leq 3$ and $0 \leq b \leq 3$. When the bids are revealed, if there was a highest bid, the person who made that bid pays the amount of the bid and receives the object. If $a = b$, a winner is chosen randomly by tossing a fair coin, and the person chosen as winner pays their bid and receives the object.

The object on sale is worth £3 for Alice and £2 for Bob.

(a) Describe this auction as a game in normal form in which players' utilities are the expected values of their profits. ***(6 marks)***

(b) Find all weakly dominated strategies and all strictly dominated strategies. ***(6 marks)***

(c) Find all pure-strategy Nash equilibria of this game. ***(5 marks)***

(ii) Alice Inc. and Bob Plc. form a duopoly in the market for kryptonite. The cost of producing 1 gram of kryptonite is £10 for Alice and £20 for Bob. The price $p(q)$ of one gram of kryptonite as a function of its supply q in grams is given by $p(q) = 1000 - q/10$. Alice and Bob will decide their production levels independently and simultaneously.

Find a production profile which results in a Nash equilibrium. ***(8 marks)***

2 (i) Alice and Bob face the following game in normal form

	l	m	r
U	1, 2	3, 0	1, 3
M	-1, 3	2, 5	0, 4
D	0, 5	7, 0	3, 2

(a) Eliminate iteratively all dominated strategies. ***(2 marks)***

(b) Find *all* Nash equilibria of this game. ***(11 marks)***

(ii) Alice and Bob face the following game

	l	r
U	0, 0	4, 4
D	2, 7	9, 3

and choose to negotiate an outcome, with the knowledge that, if they fail to strike a deal, utilities of 3 will be imposed on both.

(a) Sketch the cooperative payoff region of the game. ***(4 marks)***

(b) Describe parametrically the payoffs that satisfy the *Individual Rationality* and *Pareto Optimality* conditions. ***(3 marks)***

(c) Find the Nash Bargain of this setup. ***(5 marks)***

3 (i) Country A learns that country B is boosting its military in preparation for a possible invasion. A has a choice between preparing for an invasion, at a cost if 100 units of wealth, and not preparing, in full view of B's spies. In case of invasion, a prepared A can choose to destroy its entire infrastructure at an additional cost of 900 units of wealth, or not to do so. If an invasion occurs and A's infrastructure is not destroyed country B gains 100 units of wealth and A loses 100 units of wealth (in addition to the preparation cost of 100 units, if A has prepared). If an invasion occurs and A's infrastructure has been destroyed, B loses 5 units of wealth. If no invasion occurs, country B gets nothing.

(a) Describe this game using a tree, carefully labelling all its components. ***(5 marks)***

(b) Solve this game using backward induction. ***(3 marks)***

(c) Describe the game in strategic form, find all its pure-strategy Nash equilibria and indicate which of these is subgame perfect. ***(7 marks)***

(ii) (a) State Zermello's Theorem for finite, two-player sequential games. ***(3 marks)***

(b) Consider a game played with 10 dots drawn on a sheet of paper. Two players, Red and Blue, alternate in choosing a pair of dots which has not been chosen before by either player, and connecting the two dots with a line of their colour. Red moves first.

The game ends if either there are four dots pairwise connected with lines of same colour, in which case that colour is declared the winner, or if all pairs of points have been connected with a line, but there is no winner, in which case the game ends with a draw.

Use a strategy stealing argument to prove that there is a strategy that guarantees a win or a draw to the first player to move. ***(7 marks)***

4 (i) Consider the two-person game G given in tabular form as follows

	A	B
I	-1, -1	6, -2
II	-2, 6	3, 3

This game is played indefinitely: after each repetition of the game, the game is played again with probability $p < 1$.

(a) Describe the outcome of this game when played once. If the game were repeated $n > 1$ times and both players knew that the game is being repeated exactly n times, how would they play it? Justify your answer. ***(4 marks)***

(b) Describe a Nash equilibrium of this repeated game which results in payoffs of $-1/(1-p)$ for both players. Justify your answer. ***(2 marks)***

(c) Is Alice always playing II and Bob always playing B a Nash equilibrium of the repeated game? ***(2 marks)***

(d) Find a range of values for p for which there is a strategy profile $(\mathcal{A}, \mathcal{B})$ whose payoff for both players is the same as their payoff if they always played the strategy profile (II, B), and describe $\mathcal{A}$ and $\mathcal{B}$. Justify your answers. ***(8 marks)***

(ii) The Republic of Alice and the Kingdom of Bob share a polluted river. If the river gets decontaminated, both get a benefit of £4 billion. Alice and Bob will decide simultaneously and independently whether to go ahead with their decontamination projects, and once these are started they cannot be stopped. The cost of decontamination for Alice is £2 billion. The cost of decontamination for Bob is either £1 billion or £3 billion: Bob knows this cost but Alice only knows that Bob's cost is £1 billion with probability $p < 1/2$.

(a) Model this as a Bayesian game. ***(3 marks)***

(b) Find the Bayes-Nash equilibria of this game. ***(6 marks)***

End of Question Paper