The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

MAS377 Mathematical Biology **2 hours**

Marks will be awarded for your best ***three*** *answers.*

1 The dynamics of a prey species, X, and its predator, Y, are governed by the following ordinary differential equations,

$$\frac{dX}{dt} = X(a - bX - cY) \tag{1}$$

$$\frac{dY}{dt} = Y(\alpha - \beta Y + \gamma X) \tag{2}$$

with $a, b, c, \alpha, \beta, \gamma$ positive constants.

(i) How would you change equations (1)–(2) to make this system representative of (a) a pair of competing species and (b) a pair of co-operating species? ***(2 marks)***

(ii) The predator-prey system defined by equations (1)–(2) gives four possible equilibria: $(X^*, Y^*) \in \{(0,0), (\hat{X}, 0), (0, \hat{Y}), (\bar{X}, \bar{Y})\}$. Let $c = \beta = 1$. Derive the equations satisfied by the nullclines. Using these, sketch the two qualitatively different phase portraits this system can produce - one where all four possible equilibria occur for $X, Y \geq 0$ and one where only three occur. In each case, clearly show the nullclines, equilibria, qualitative directions of flow and two sample trajectories. ***(10 marks)***

(iii) The $(0,0)$ equilibrium is always unstable. Considering the two phase portraits from part (ii), which of the other equilibria also appears to never be stable? Using stability analysis on this equilibrium, prove that it is indeed never stable. ***(6 marks)***

(iv) Now assume that the rate of predation saturates when the prey density is high. Taking some numeric values for all of the parameters, the revised system is now given by,

$$\frac{dX}{dt} = X\left(2 - X - \frac{Y}{X+2}\right) \tag{3}$$

$$\frac{dY}{dt} = Y\left(1 - Y + \frac{X}{X+2}\right) \tag{4}$$

Derive the equations satisfied by the nullclines in this updated system. Using these, sketch the phase portrait for the new system in equations (3)-(4), showing nullclines, equilibria and qualitative directions of flow. *Sample trajectories are not required.* Explain why the assumption of saturating predation in equations (3)-(4) is more realistic than the linear predation in (1)–(2). ***(7 marks)***

2 A population of plants is exposed to an infectious disease. Partitioning the population into either susceptible (S) or infected (I) compartments, the dynamics of this population are given by the ordinary differential equations,

$$\frac{dS}{dt} = (b - q(S+I))\,S - \beta SI - dS \qquad (5)$$

$$\frac{dI}{dt} = \beta SI - (\alpha + d)I \qquad (6)$$

for b, q, d, β and α positive constants, with $b > d$.

(i) What biological process does q represent? ***(2 marks)***

(ii) This model yields a zero equilibrium, a 'disease-free' equilibrium and an 'endemic' equilibrium. Find the values of $\hat{S}$ and $\hat{I}$ at the disease-free equilibrium, and show that,

$$S^* = \frac{\alpha + d}{\beta} \text{ and } I^* = \frac{b - qS^* - d}{q + \beta}$$

at the endemic equilibrium. ***(6 marks)***

(iii) (a) Find the Jacobian matrix, J, for this system. Thus show that the disease-free equilibrium loses stability when $R_0 \equiv \beta\hat{S}/(\alpha + d) > 1$. ***(6 marks)***

(b) *Without substituting in the equilibrium values, S^* and I^**, show that at the endemic equilibrium the Jacobian yields,

$$tr(J) = -qS^* \text{ and } det(J) = \beta S^* I^* (q + \beta).$$

What can you conclude about the stability of the endemic equilibrium? ***(5 marks)***

(iv) Let $b = 2$, $d = 1$, $\beta = 1$ and $\alpha = 1$. Sketch a bifurcation diagram, plotting equilibrium values for I on the vertical axis and varying q from 0 to 1 on the horizontal axis. Use solid lines for stable equilibria and dashed lines for unstable equilibria. At what value of q does a bifurcation occur, and what type of bifurcation is it? ***(6 marks)***

3 The regulated transcription of a gene is represented by the differential equation

$$\frac{dm}{dt} = -\mu m + f(t), \quad t \geq 0, \tag{7}$$

where $m(t)$ represents the concentration of the mRNA transcript associated with the gene, and μ is a positive constant.

(i) What are the meanings of the parameter μ and the function $f(t)$?
(2 marks)

(ii) Why should $f(t)$ be non-negative and bounded above? ***(2 marks)***

(iii) Consider the following form for $f(t)$:

$$f(t) = \begin{cases} A, & 0 \leq t \leq T \\ Ae^{-\nu(t-T)}, & t > T \end{cases}$$

where A and ν are positive constants. Sketch $f(t)$. By solving (7), show that if $m(0) = 0$, then

$$m(t) = \frac{A}{\mu}\left(1 - e^{-\mu t}\right), \quad 0 \leq t \leq T.$$

(5 marks)

(iv) Using the expression for $m(T)$ obtained from (iii) as initial condition, and assuming that $\mu \neq \nu$, solve (7) to show that

$$m(t) = \frac{A}{\mu}\left[\frac{\mu}{\mu - \nu}e^{\nu(T-t)} - \frac{\nu}{\mu - \nu}e^{\mu(T-t)} - e^{-\mu t}\right], \quad t > T.$$

(6 marks)

(v) By differentiating (7) with respect to t, show that if $m(t)$ has a turning point at a value of t at which $f(t)$ is decreasing, then it is a local maximum. Show that $m(t)$ reaches its maximum value at time

$$\tau = \frac{1}{\mu - \nu}\ln\left[e^{(\mu-\nu)T} + \frac{\mu - \nu}{\nu}e^{-\nu T}\right].$$

Calculate τ when $\mu = 0.03\text{min}^{-1}$, $\nu = 0.01\text{min}^{-1}$ and $T = 30\text{min}$.
(10 marks)

4 A model for the expression of an auto-repressive gene is given by

$$\frac{dM}{dt} = -\mu M + f[P(t-\tau_1)], \qquad (8)$$

$$\frac{dP}{dt} = M(t-\tau_2) - \mu P, \qquad (9)$$

where $M(t)$ and $P(t)$ represent the concentrations of the mRNA and protein corresponding to the gene, respectively, τ_1 and τ_2 are non-negative constants, μ and ν are positive constants, and $f(P)$ is a monotonic decreasing function with $f(0) = 1$ and $\lim_{P\to\infty} f(P) = 0$.

(i) Explain the biological meaning of τ_1 and τ_2. ***(2 marks)***

(ii) Show graphically that the model always has a unique positive steady state $M = M_*, P = P_*$ and show that P_* satisfies the equation $\mu^2 P_* = f(P_*)$. ***(4 marks)***

(iii) Sketch the phase portrait for the model when $\tau_1 = \tau_2 = 0$. By considering the Jacobian matrix in this case, show that the steady state is a stable spiral. Sketch the trajectory starting at the point $(M = 0, P = 0)$. ***(6 marks)***

(iv) Show that the model can be written as a single second-order equation for P:

$$\frac{d^2P}{dt^2} + 2\mu\frac{dP}{dt} + \mu^2 P = f[P(t-\tau)],$$

where $\tau = \tau_1 + \tau_2$.
By setting $P(t) = P_* + p_0 e^{\lambda t}$ (p_0 small) and linearising this equation, show that λ satisfies

$$(\lambda + \mu)^2 = \phi e^{-\lambda\tau}, \quad \phi < 0, \qquad (10)$$

and explain what ϕ is. ***(7 marks)***

(v) Show that if Eq. (10) can have a pure imaginary solution $\lambda = i\omega$, $\omega \in \mathbb{R}$, then

$$\mu^2 - \omega^2 = \phi\cos\omega\tau \quad \text{and} \quad 2\mu\omega = -\phi\sin\omega\tau.$$

Show that a necessary condition for this to be possible is that $-\phi > \mu^2$. If this condition is met, show that $\omega^2 = -\phi - \mu^2$ and that τ must take the value

$$\tau = \tau_B \equiv \frac{1}{\omega}\tan^{-1}\left(\frac{-2\mu\omega}{\mu^2 - \omega^2}\right).$$

What kind of bifurcation occurs as τ increased through the value τ_B? What behaviour do you expect the model to exhibit for $\tau > \tau_B$? ***(6 marks)***

End of Question Paper