The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2018–19

Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers.
The maximum possible mark for the paper is 100.

1 (a) Second order linear partial differential equations for $u(x, y)$ can be written in the general form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F and G are functions of x and y.

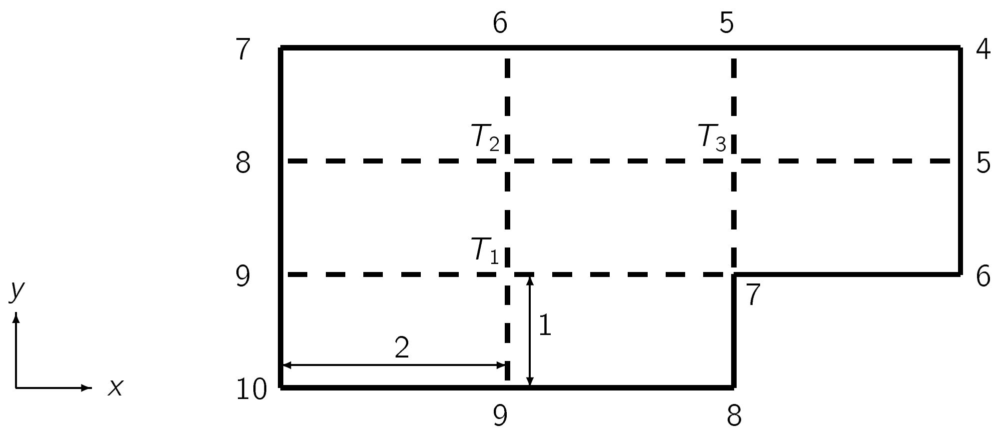
(i) What are the conditions on A, B, C, D, E, F and G which determine whether the equation is elliptic, parabolic or hyperbolic?

(3 marks)

(ii) Identify the regions of the (x, y) plane where the equation

$$\frac{\partial^2 u}{\partial x^2} + (x-1)\frac{\partial^2 u}{\partial x \partial y} = 0$$

is elliptic, parabolic or hyperbolic. ***(4 marks)***



 Question 1 continued on next page

(b) $T(x, y)$ satisfies the boundary conditions indicated in the figure, and the differential equation

$$x^2 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

The figure (where the lower left corner is the origin $x = y = 0$) shows the solution domain, divided into intervals of length $\Delta x = 2$ in the x direction, and length $\Delta y = 1$ in the y direction.

(i) In the interior of the region is this differential equation elliptic, parabolic or hyperbolic? ***(2 marks)***

(ii) Use the finite difference formulae on the formula sheet to find the finite difference equations required to find estimates of the nodal values T_1, T_2 and T_3. ***(8 marks)***

(iii) Express the finite difference equations in part (ii) in the form $A\mathbf{T} = \mathbf{b}$, where A is a 3×3 matrix, $\mathbf{T} = [T_1, T_2, T_3]^T$ and $\mathbf{b} = [25, 14, 16]^T$, where you should give the matrix A.

Hence, using Gaussian elimination or otherwise, show that

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}.$$

(8 marks)

2 The temperature $T(x,t)$ satisfies the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant 1). \tag{1}$$

(a) If $T_{i,j} = T(x_i, t_j)$, with $i = 0$ and $i = N$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the implicit scheme

$$-(k-\beta)T_{i+1,j} + (1+2k)T_{i,j} - (k+\beta)T_{i-1,j} = T_{i,j-1}$$

for $i = 1, \dots, N-1$ and $j = 1, 2, \dots,$ where

$$k = \frac{\Delta t}{(\Delta x)^2} \quad \text{and} \quad \beta = \frac{\Delta t}{2\Delta x}.$$ ***(5 marks)***

(b) Equation (1) is to be solved (approximately) over $0 \leqslant x \leqslant 1$, with boundary conditions $T(0,t) = 10$ and $T(1,t) = 20$, and initial temperature distribution $T(x,0) = 10 + 10x$.

Taking $\Delta x = 0.2$ and $\Delta t = 0.1$, use the implicit scheme in part (a) to write down the system of equations for the temperature at $x = 0.2, 0.4, 0.6, 0.8$ and time $t = 0.1$. (Note that you **do not** need to solve the equations.)

(12 marks)

(c) Show from your answer to part (b) that the Jacobi iteration equations to find the $(k+1)$th iteration from the kth iteration are

$$\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{bmatrix}^{(k+1)} = \frac{1}{6}\begin{bmatrix} 39.5 \\ 14 \\ 16 \\ 63 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 0 & 9 & 0 & 0 \\ 11 & 0 & 9 & 0 \\ 0 & 11 & 0 & 9 \\ 0 & 0 & 11 & 0 \end{bmatrix}\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{bmatrix}^{(k)}$$ ***(8 marks)***

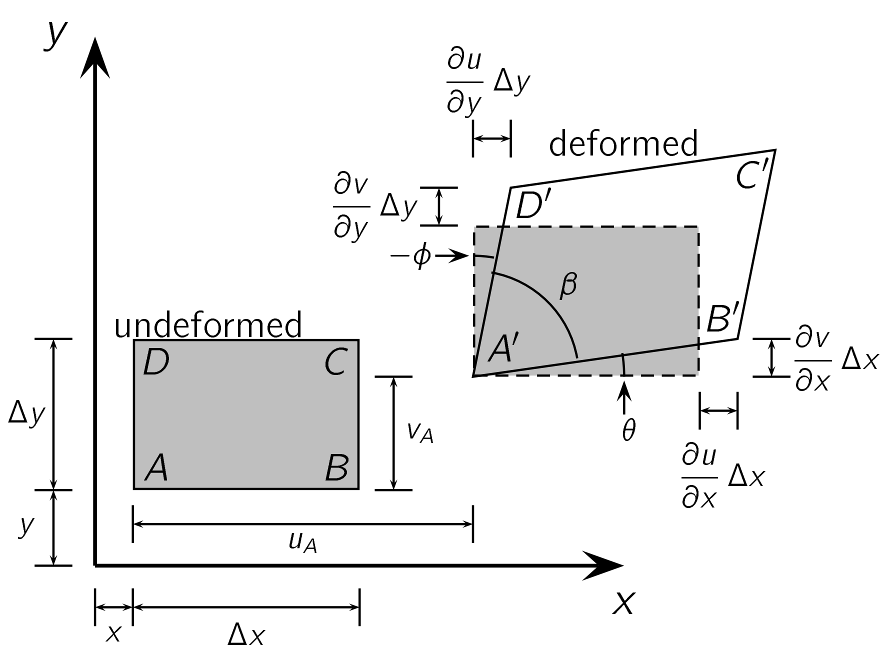
3 (a) A rectangular surface on a positive x-plane is defined by $x = 0$, $0 \leqslant y \leqslant \pi$ and $0 \leqslant z \leqslant 2$ (where all lengths are measured in m). The stress tensor depends on y and is given by (in units of Pa)

$$[\sigma] = \begin{bmatrix} \sin y & y^2 & \cos y \\ y^2 & e^y & y \\ \cos y & y & 1 \end{bmatrix}.$$

Show that the total stress force $\mathbf{f}$ on the rectangular surface is

$$\mathbf{f} = \frac{2}{3}\begin{bmatrix} 6 \\ \pi^3 \\ 0 \end{bmatrix} \text{N}.$$ ***(6 marks)***

(b) Referring to the figure, for small displacements u ($= u_A$ at point A) in the x-direction, and v ($= v_A$ at point A) in the y-direction, define the normal strain ε_{xx} and the engineering shear strain γ_{xy}. ***(3 marks)***



Hence show that

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \text{and} \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$ ***(12 marks)***

If w is the small displacement in the z-direction, what would you expect the normal strain ε_{zz} and the engineering shear strain γ_{xz} to equal? ***(2 marks)***

 Question 3 continued on next page

(c) The matrix relating the *engineering* strains to the stresses for an isotropic material is given by

$$\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where λ and μ are constants.

What is the matrix relating the *tensorial* strains to the stresses for an isotropic material? ***(2 marks)***

4 An incompressible fluid of constant density ρ is contained in a channel between horizontal plane solid boundaries at $y = 0$ and $y = a$, where a is a positive constant. The fluid is in steady motion. The boundary at $y = 0$ is stationary, and the boundary at $y = a$ moves with constant speed U in the x-direction. Assume that there is no variation in the z-direction.

The fluid velocity $\mathbf{v}$ is given by

$$\mathbf{v} = u(y)\,\mathbf{i},$$

where $\mathbf{i}$ is the unit vector in the x-direction. $\mathbf{v}$ satisfies

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{v},$$

where μ is the constant viscosity.

(a) Show that

$$(\mathbf{v}.\nabla)\mathbf{v} = \mathbf{0},$$

and hence that

$$\mathbf{0} = -\nabla p + \mu\frac{d^2u}{dy^2}\,\mathbf{i}.$$

(5 marks)

Show that

$$u(y) = U\frac{y}{a} + \frac{Ga^2}{2\mu}\frac{y}{a}\left(1 - \frac{y}{a}\right)$$

for some constant G. ***(12 marks)***

(b) If the fluid velocity at the midpoint of the channel is $\frac{3}{4}U$ in the x-direction, find u. ***(4 marks)***

Find the fluid flux $\int_0^a u\,dy$ across a plane perpendicular to the flow, per unit distance in the z-direction. Is this flux greater or less than that which would result if the fluid velocity were U throughout the channel?

(4 marks)

5 (a) In index notation the i-component of the vector product of $\mathbf{u}$ and $\mathbf{v}$ may be expressed as

$$(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k,$$

and the i-component of the curl of a vector field $\mathbf{F}$ may be expressed as

$$(\nabla \times \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j},$$

where ε_{ijk} is the Levi-Civita tensor.

To answer the questions below you may assume that

$$\varepsilon_{kij}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}.$$

(i) Show, using index notation, that

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u}).$$

(11 marks)

(ii) If

$$\boldsymbol{\omega} = \nabla \times \mathbf{v},$$

use index notation to show that

$$\mathbf{v} \times \boldsymbol{\omega} = \nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

(8 marks)

(b) Let

$$\mathbf{F} = \phi \mathbf{a},$$

where ϕ is a scalar field and $\mathbf{a}$ is an arbitrary constant vector. By applying Gauss' theorem (see the formula sheet) to $\mathbf{F}$, show that

$$\int_V \nabla \phi \, dV = \int_S \phi \hat{\mathbf{n}} \, dS.$$

(6 marks)

End of Question Paper

Formula Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,\,j+1} - U_{i,\,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,\,j} - U_{i,\,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,\,j} - U_{i,\,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,\,j} - U_{i-1,\,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\ t_j) \approx \frac{U_{i+1,\,j} - 2U_{i,\,j} + U_{i-1,\,j}}{(\Delta x)^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K - \frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.

Gauss' theorem

Let $\mathbf{F}(x, y, z)$ be a vector field. Consider a closed volume V bounded by surface S. At each point on S let the outward normal be given by the unit vector $\hat{\mathbf{n}}$. Then *Gauss' theorem* (or the *divergence theorem*) is:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$