Data provided: Formula sheet

MAS381

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Mathematics III (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The circle C passes through the points $A(2,1)$, $B(0,5)$ and $C(-1,2)$.

(a) Assuming that the three points are in the real plane, find the equation of the circle in the form $(x-x_0)^2+(y-y_0)^2 = R^2$, where (x_0, y_0) are the coordinates of the centre of the circle and R is the radius.

(b) Assuming that the three points are in the complex plane, write the equation of the circle in the form $|z - z_0| = R$ and $z = z_0 + Re^{i\phi}$, respectively. ***(11 marks)***

(ii) Let us consider the function $u(x, y) = x^3 - 3xy^2 - 5y$. Show that $u(x, y)$ is a harmonic function and find the harmonic conjugate, $v(x, y)$, of the function $u(x, y)$. Determine any constants involved in the solution taking into account that the function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ takes the value $1 + i$ when $z = 1$. ***(14 marks)***

2

(i) For each of the vector fields, $\mathbf{F}$, calculate $\nabla \times \mathbf{F}$

(a) $\mathbf{F} = (x^2 + yz)\mathbf{i} + xz\mathbf{j} - xy\mathbf{k}$,

(b) $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + (xy + 4z)\mathbf{k}$.

(4 marks)

(ii) Which of the two vector fields can be written as $\mathbf{F} = \nabla U$, where U is a scalar function? For this vector determine the most general form of U. ***(8 marks)***

(iii) Let C be the circle of radius one in the xy plane, centered on zero. Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector (a). (N.B. All contour integrals should be evaluated in the *counterclockwise* direction). ***(13 marks)***

3 (i) Use the complex integration technique and the residue theorem to determine the value of the real integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+2)}.$$

(15 marks)

(ii) Evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (xy, 0, -yz)$ and C is the line segment connecting the points $(-1, 2, 0)$ and $(3, 0, 1)$ ***(10 marks)***

4 (i) Expand the function

$$f(z) = \frac{3i}{(2z+1)(z+i)},$$

into a Laurent series in the interval $1/2 \le |z| \le 1$. ***(15 marks)***

(ii) Let D be the disc of radius a centred at $(0, 0)$, and let $\mathbf{u}$ be the vector field $(xy^2, 0)$. Let C be the boundary curve of D. Verify the divergence theorem

$$\iint_D div(\mathbf{u}) dA = \int_C \mathbf{u} \cdot d\mathbf{n}$$

in this case. ***(10 marks)***

End of Question Paper

Formula sheet

- The general formula for the residue at a pole z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}; \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

$$\sin m\theta \cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]$$

- The polar and spherical area elements are given by

$$dA = r dr d\theta, \quad dA = r^2 \sin\phi \, d\phi \, d\theta$$